

Stability of Ginzburg-Landau Pulses Via Fredholm Determinants of Birman-Schwinger Operators

John Zweck

New York Institute of Technology

<https://drjohnzweck.github.io>

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Joint with **Erika Gallo** and Yuri Latushkin

NEW YORK INSTITUTE
OF TECHNOLOGY



The Big Picture

- Stability of stationary pulses characterized by Evans det, $E(\lambda)$.
- Computation of $E(\lambda)$ requires solution of stiff ODE system
 - Challenging for high dimensional problems
- Can we compute point spectrum for **breathers**??
- Breathers give rise to ∞ -dim ODE system (via Fourier series)
 - Theoretical definition of $E(\lambda)$ may not even be possible!

Alternate Approach:

Via a Fredholm determinant of a trace class operator

Today: **Stationary** Pulses + Low-Dimensions only!

In case of **Complex Ginzburg-Landau Equation**

Cubic-quintic complex Ginzburg-Landau

$$i\partial_t\psi + \left(\frac{D}{2} - i\beta\right)\partial_x^2\psi + (\alpha - i\delta)\psi + [\gamma - i\epsilon + (\nu - i\mu)|\psi|^2] |\psi|^2\psi = 0.$$

Reformulate as a system for $\psi = [\text{Re}(\psi) \quad \text{Im}(\psi)]^T$

$$\partial_t\psi = (\mathbf{B}\partial_x^2 + \mathbf{N}_0 + \mathbf{N}_1|\psi|^2 + \mathbf{N}_2|\psi|^4)\psi,$$

where

$$\mathbf{B} = \begin{bmatrix} \beta & -\frac{D}{2} \\ \frac{D}{2} & \beta \end{bmatrix}, \quad \text{assume } (\beta, D) \neq (0, 0),$$

and

$$\mathbf{N}_0 = \begin{bmatrix} \delta & -\alpha \\ \alpha & \delta \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} \epsilon & -\gamma \\ \gamma & \epsilon \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}.$$

Linearization of CQ-CGLE

- Linearizing CQ-CGLE about ψ , we obtain

$$\partial_t \mathbf{u} = \mathcal{L} \mathbf{u},$$

$$\mathcal{L} = \mathcal{L}_\infty + \mathbf{M}, \quad \text{with} \quad \mathcal{L}_\infty = \mathbf{B} \partial_x^2 + \mathbf{N}_0,$$

where

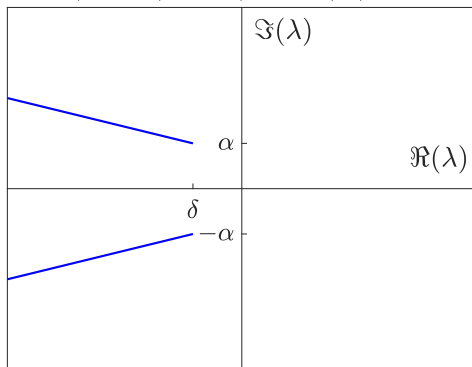
$$\mathbf{M}(x) = \mathbf{N}_1 |\psi|^2 + \mathbf{N}_2 |\psi|^4 + (2\mathbf{N}_1 + 4\mathbf{N}_2 |\psi|^2) \psi \psi^T.$$

- Linear stability of the ψ is determined by

$$\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \cup \sigma_{\text{pt}}(\mathcal{L}).$$

The essential spectrum of $\mathcal{L}$

$$\lambda = (\delta \pm i\alpha) - \mu^2(\beta \pm iD/2), \quad \mu \in \mathbf{R}$$



Stable for $\delta < 0$, $\beta > 0$

Goal: Find a rigorous way to calculate the point spectrum.

Trace class & Hilbert Schmidt operators; Fredholm dets

$$\mathcal{B}_p(\mathcal{H}) = \{\mathcal{K} : \mathcal{H} \rightarrow \mathcal{H} \text{ is compact} : \sum_{j=1}^{\infty} |\mu_j|^p < \infty\},$$

where $\{\mu_j\}_{j=1}^{\infty}$ are the singular values of $\mathcal{K}$.

- $\mathcal{B}_1(\mathcal{H}) = \text{Trace class} \subsetneq \mathcal{B}_2(\mathcal{H}) = \text{Hilbert Schmidt operators}$
- Every $\mathcal{K} \in \mathcal{B}_2(L^2(\mathbb{R}, \mathbb{C}^k))$ has a kernel $\mathbf{K}(x, y) \in \mathbb{C}^{k \times k}$.
- For $\mathcal{K} \in \mathcal{B}_1(\mathcal{H})$ with $\{\eta_k\}$ an ONB for $\mathcal{H}$:

$$\text{Tr}(\mathcal{K}) := \sum \langle \eta_k, \mathcal{K} \eta_k \rangle = \sum \lambda_i(\mathcal{K}) = \int_{\mathbb{R}} \text{Tr}(\mathbf{K}(x, x)) dx$$

$$\prod_i^{\infty} (1 + \lambda_i) = \sum_{n=0}^{\infty} \sum_{i_1 < \dots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} = \sum_{n=0}^{\infty} \text{Tr}(\wedge^n \mathcal{K}) =: \det_1(\mathcal{I} + \mathcal{K}).$$

- If $\mathcal{K} \in \mathcal{B}_1(\mathcal{H})$, then $\det_2(\mathcal{I} + \mathcal{K}) = e^{-\text{Tr}(\mathcal{K})} \det_1(\mathcal{I} + \mathcal{K})$.

Birman-Schwinger Program for Point Spectrum¹

- We convert differential operator, $\mathcal{L} - \lambda$ to a Green's operator, $\mathcal{K}(\lambda)$.
- We show that $\mathcal{K}(\lambda)$ is a trace class operator.
- Therefore, $\mathcal{K}(\lambda)$ has a Fredholm determinant and

$$\lambda \in \sigma_{\text{pt}}(\mathcal{L}) \iff \mathcal{I} + \mathcal{K}(\lambda) \text{ is invertible} \iff \det_1(\mathcal{I} + \mathcal{K}(\lambda)) = 0.$$

Pros:

- Don't have to solve stiff systems of differential equations
- For stationary pulses: $E(\lambda) = \det_1(\mathcal{I} + \mathcal{K}(\lambda))$

Cons:

- Fredholm determinant is of an infinite-dimensional operator

BUT

Bornemann devised a method to compute $\det_1(\mathcal{I} + \mathcal{K})$.

¹Gesztesy, Latushkin, & Makarov, *Arch. Rational Mech. Anal.*, 2007

Conversion to First-Order System

To calculate the spectrum of $\mathcal{L}_\infty$, we convert the system $(\mathcal{L}_\infty - \lambda)\mathbf{u} = 0$ to the first-order **unperturbed** problem

$$\partial_x \mathbf{v}_\infty = \mathbf{A}(\lambda) \mathbf{v}_\infty, \quad \mathbf{v}_\infty = [\mathbf{u} \ \mathbf{u}_x]^T,$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \\ \mathbf{B}^{-1}(\lambda - \mathbf{N}_0) & \mathbf{0}_{2 \times 2} \end{bmatrix} \in \mathbb{C}^{4 \times 4}.$$

Similarly, we convert $(\mathcal{L} - \lambda)\mathbf{u} = 0$ to the **perturbed** problem

$$\partial_x \mathbf{v} = (\mathbf{A}(\lambda) + \mathbf{R}(x)) \mathbf{v}, \quad \mathbf{v} = [\mathbf{u} \ \mathbf{u}_x]^T,$$

where the perturbation $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{B}^{-1} \mathbf{M} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{B}^{-1} \mathbf{M} |\mathbf{M}|^{-1/2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} |\mathbf{M}|^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{R}_\ell \mathbf{R}_r.$$

The Unperturbed System

Define $\mathcal{L}_A : H^1(\mathbb{R}, \mathbb{C}^k) \rightarrow L^2(\mathbb{R}, \mathbb{C}^k)$ [$k = 4$] by

$$\mathbf{w} = \mathcal{L}_A \mathbf{v} := -\partial_x \mathbf{v} + \mathbf{A}(\lambda) \mathbf{v}.$$

$$\lambda \in \sigma_{\text{ess}}(\mathcal{L}) \iff \mathcal{L}_A(\lambda) \text{ is not invertible.}$$

For $\lambda \notin \sigma_{\text{ess}}(\mathcal{L})$, the Green's operator $\mathcal{G}_A = \mathcal{L}_A^{-1}$ is given by

$$\mathbf{v}(x) = (\mathcal{G}_A \mathbf{w})(x) = \int_{\mathbb{R}} \mathbf{K}_A(x, y) \mathbf{w}(y) dy,$$

where

$$\mathbf{K}_A(x, y) = \Psi(x - y),$$

with

$$\Psi(x) = \begin{cases} -e^{\mathbf{A}x} \mathbf{Q}, & x \geq 0, \\ e^{\mathbf{A}x} (\mathbf{I} - \mathbf{Q}), & x \leq 0, \end{cases}$$

where $\mathbf{Q}$ is the $k \times k$ projection matrix onto the stable eigenspace of $\mathbf{A}$.

The Perturbed System and Birman-Schwinger Trick

$\lambda \in \sigma_{\text{pt}}(\mathcal{L}) \iff \exists \mathbf{v} \in L^2(\mathbb{R}, \mathbb{C}^k)$ which solves the perturbed system

$$\partial_x \mathbf{v} = (\mathbf{A}(\lambda) + \mathbf{R}(x))\mathbf{v}$$

or

$$[\mathcal{L}_A + \mathbf{R}(x)] \mathbf{v} = 0$$

Birman-Schwinger Trick:

$$\mathcal{L}_A + \mathbf{R} = \mathcal{L}_A[\mathbf{I} + \mathcal{L}_A^{-1}\mathbf{R}] = \mathcal{L}_A[\mathbf{I} + \mathcal{G}_A\mathbf{R}_\ell\mathbf{R}_r],$$

So, for $\lambda \notin \sigma_{\text{ess}}(\mathcal{L})$,

$$\lambda \in \sigma_{\text{pt}}(\mathcal{L}) \iff \mathcal{L}_A + \mathbf{R} \text{ is not invertible}$$

$$\iff \mathbf{I} + \mathcal{G}_A(\lambda)\mathbf{R}_\ell\mathbf{R}_r \text{ is not invertible}$$

$$\iff \mathcal{I} + \mathcal{K}(\lambda) := \mathbf{I} + \mathbf{R}_r\mathcal{G}_A(\lambda)\mathbf{R}_\ell \text{ is not invertible}$$

Explicit Formula for the Green's Kernel

The operator $\mathcal{K}$ is given by

$$\mathcal{K}\phi(x) = \int_{\mathbb{R}} \mathbf{K}(x, y)\phi(y)dy.$$

with kernel

$$\mathbf{K}(x, y) = \begin{cases} -\mathbf{R}_r(x)\mathbf{Q}e^{\mathbf{A}(x-y)}\mathbf{Q}\mathbf{R}_\ell(y), & x \geq y, \\ \mathbf{R}_r(x)(\mathbf{I} - \mathbf{Q})e^{\mathbf{A}(x-y)}(\mathbf{I} - \mathbf{Q})\mathbf{R}_\ell(y), & x < y, \end{cases}$$

Theorem

- $\mathcal{K}$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}, \mathbb{C}^k)$
- $\mathbf{K} \in L^2(\mathbb{R}, \mathbb{C}^{k \times k}) \cap C^0(\mathbb{R}, \mathbb{C}^{k \times k})$.

Proof.

$\mathbf{K}$ is continuous, as $\mathbf{R}_r\mathbf{R}_\ell = \mathbf{0}$. □

General Criteria for being Trace Class⁴

Theorem (ZGL)

Let $\mathcal{K} \in \mathcal{B}_2(L^2(\mathbb{R}, \mathbb{C}^k))$ be given in terms of an L^2 -kernel $\mathbf{K}$ such that

$$(\mathcal{K}\phi)(x) = \int_{\mathbb{R}} \mathbf{K}(x, y)\phi(y)dy,$$

where $\mathbf{K}(x, y)$ is Lipschitz-continuous and $\mathbf{K}$ and its first partial derivatives decay exponentially almost everywhere. Then $\mathcal{K} \in \mathcal{B}_1(L^2(\mathbb{R}, \mathbb{C}^k))$.

- Fredholm² and Weidmann³ showed this for scalar kernels on $[-1, 1]$.
- We generalize this to matrix-valued kernels on $\mathbb{R}$.
- And to operator-valued kernels, $\mathbf{K} \in C^{0,1}([-1, 1] \times [-1, 1], \mathcal{B}_2(H))$.

²Fredholm, *Acta Mathematica*, 1903

³Weidmann, *Mathematische Annalen*, 1966

⁴Zweck, Gallo, Latushkin, *Bol. Soc. Mat. Mex.*, 2025

Lipschitz Continuity of the CQ-CGLE Kernel

Theorem

Suppose that $\psi \in C^1(\mathbb{R}, \mathbb{R}^2)$, ψ, ψ_x decay exponentially, $|\psi| > 0$, on $\mathbb{R}$ and $\epsilon, \gamma \geq 0$. Suppose

$\mathbf{a} = [\epsilon \ \gamma]^T$ and $\mathbf{b} = [\mu \ \nu]^T$ are not both $\mathbf{0}$, and

if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then $\max_{x \in \mathbb{R}} |\psi(x)|^2 < r$ for certain $r = r(\mathbf{a}, \mathbf{b}) \leq \infty$.

Then $\mathbf{K}$ is Lipschitz continuous.

Proof: By Cauchy's Integral Formula

$$|\mathbf{M}(x)| = \sqrt{(\mathbf{M}^*(x)\mathbf{M}(x))} = \frac{1}{2\pi i} \int_{\gamma} \sqrt{z}(\mathbf{M}^*(x)\mathbf{M}(x) - z)^{-1} dz.$$

- Assumptions imply that $\det(\mathbf{M}(x)) \neq 0$ and so $x \mapsto |\mathbf{M}(x)|^{\pm 1/2}$ is C^1 .
- So, $\mathbf{K}(x, y) \in C^{0,1}(\mathbb{R}, \mathbb{C}^{k \times k})$, using FTC across diagonal.

Analytical Formula for Fredholm Determinant

Theorem (GZL)

Let $\mathcal{K} \in \mathcal{B}_1(L^2(\mathbb{R}, \mathbb{C}^k))$ have continuous exponentially-decaying kernel. Let

$$K_{\mathbf{j}}^{(n)}(x_1, \dots, x_n) := \det [\mathbf{K}_{j_\alpha j_\beta}(x_\alpha, x_\beta)]_{\alpha, \beta=1}^n,$$

where

$$\mathbf{j} \in J_k^{(n)} = \{(j_1 \dots j_n) : 1 \leq j_\ell \leq k \ \forall \ell\}.$$

Then

$$\det_1(\mathcal{I} + \mathcal{K}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathbf{j} \in J_k^{(n)}} \int_{\mathbb{R}^n} K_{\mathbf{j}}^{(n)}(\mathbf{x}) d\mathbf{x}.$$

Calculating $\det_1(\mathcal{I} + \mathcal{K})$ seems hopeless!?

Approximating the Determinant⁵

To approximate

$$\det_1(\mathcal{I} + \mathcal{K}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathbf{j} \in J_k^{(n)}} \int_{\mathbb{R}^n} K_{\mathbf{j}}^{(n)}(\mathbf{x}) d\mathbf{x},$$

- **Truncate** $\mathbb{R}$ to $[-L, L]$.
- Apply a **quadrature rule** to integrate over $[-L, L]^n$.
- $|\text{Total Error}| \leq |\text{Truncation Error}| + |\text{Quadrature Error}|$

**This approximation of the operator determinant
is given by matrix determinant.**

⁵[Bornemann, 2008]

Approximation by Matrix Determinant

Let $Q = Q_M$ be a quadrature rule, evaluated at M nodes $-L \leq x_1 < x_2 < \dots < x_M \leq L$ and using positive weights $w_1, \dots, w_M$.

Define $\mathbf{K}_Q \in \mathbb{C}^{kM \times kM}$ to be the $M \times M$ block matrix

$$\mathbf{K}_Q = \begin{bmatrix} w_1 \mathbf{K}(x_1, x_1) & w_2 \mathbf{K}(x_1, x_2) & \dots & w_M \mathbf{K}(x_1, x_M) \\ w_1 \mathbf{K}(x_2, x_1) & w_2 \mathbf{K}(x_2, x_2) & \dots & w_M \mathbf{K}(x_2, x_M) \\ \vdots & & & \vdots \\ w_1 \mathbf{K}(x_M, x_1) & w_2 \mathbf{K}(x_M, x_2) & \dots & w_M \mathbf{K}(x_M, x_M) \end{bmatrix},$$

where $\mathbf{K}(x, y)$ is the $k \times k$ submatrix given by the kernel.

Then the discrete determinants are given by

$$d_{Q,1} := \det[\mathbf{I}_{kM \times kM} + \mathbf{K}_Q]$$

$$d_{Q,2} := e^{-\text{Tr}(\mathbf{K}_Q)} \det[\mathbf{I}_{kM \times kM} + \mathbf{K}_Q]$$

Theorem

Suppose that $\mathcal{K} \in \mathcal{B}_1(L^2(\mathbb{R}, \mathbb{C}^k))$ has a kernel whose elements satisfy

- ① $K_{ij} \in \text{Lip}([-L, L] \times [-L, L], \mathbb{C})$.
- ② $\exists C, a > 0$ such that $|K_{ij}(x, y)| \leq Ce^{-a(|x|+|y|)} \quad \forall x, y \in \mathbb{R}$.

Then ($p = 1, 2$)

$$|\det_p(\mathcal{I} + \mathcal{K}) - \det_p(\mathcal{I} + z\mathcal{K}|_{[-L, L]})| \leq C_1 e^{-aL}.$$

Truncation Error decreases exponentially as $L \rightarrow \infty$.

Quadrature Error (Composite Simpson's)⁶

Theorem

Let $\mathcal{K} \in \mathcal{B}_p(L^2([-L, L], \mathbb{C}^k))$ have Lipschitz kernel, $\mathbf{K}$. Let Q denote the *composite Simpson's* quadrature rule with spacing Δx . Then

$$|d_{Q,p} - \det_p(\mathcal{I} + \mathcal{K})| \leq C_2 \Delta x.$$

For many CQ-CGLE pulses $\mathbf{K}$ is Lipschitz,

and so $\det_p(\mathcal{I} + \mathcal{K})$ can be calculated with

$$|\mathbf{Total\ Error}| \leq C_1 e^{-aL} + C_2 \Delta x.$$

⁶Result extends to *adaptive* composite Simpson's

Truncation and Quadrature Error

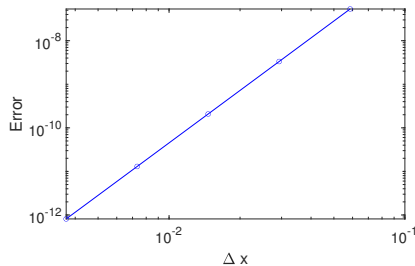
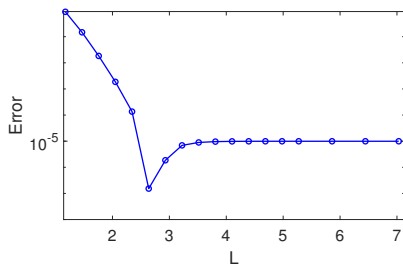


Figure: Truncation Error (Left), Quadrature Error of $O(\Delta x^4)$ (Right)

The Evans Function and the Fredholm Determinant

The kernel of the sech solution of NLSE is trace class. Therefore,⁷

$$\det_1(\mathcal{I} + \mathcal{K}(\lambda)) = E(\lambda),$$

where the Evans function is given by⁸

$$E(\lambda) = \frac{16\lambda^4}{(1 + \sigma_+)^4(1 + \sigma_-)^4},$$

with $\sigma_{\pm} = \sqrt{1 \pm 2i\lambda}$.

**We validate our approximation of $\det_1(\mathcal{I} + \mathcal{K})$
by comparison with this Evans function.**

⁷Gesztesy, Latushkin, & Makarov, *Arch. Rational Mech. Anal.*, 2007

⁸See Kapitula and Promislow

Sech Pulse: Evans vs. Fredholm Det

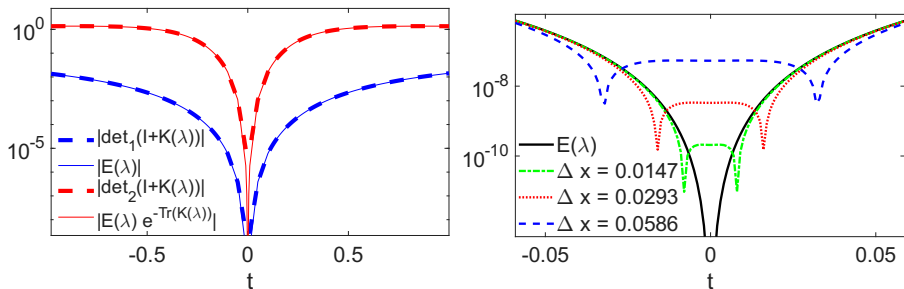


Figure: **Behaviour near order four zero at $\lambda = 0$.** **Left:** Slice along real axis. **Right:** Zoomed in version of results on the left with finer discretizations, Δx , across the pulse.

Validation Results for Sech and CGLE

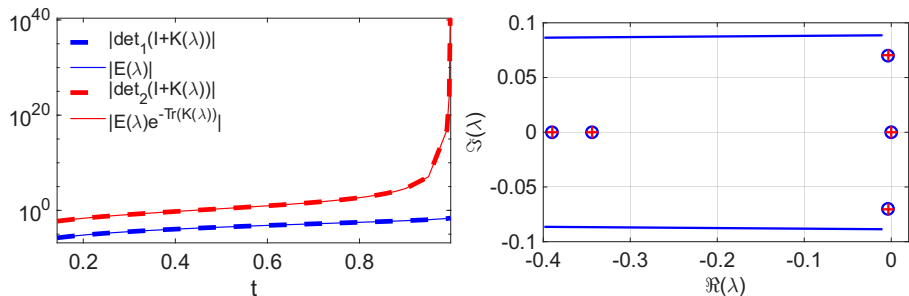


Figure: **Left:** Results for **sech pulse** as λ approaches the edge of $\sigma_{\text{ess}}(\mathcal{L})$ along imaginary axis. **Right:** Spectrum of a **stationary solution of CGLE**: Essential spectrum (solid blue lines) and eigenvalues using $\det_1(\mathcal{I} + \mathcal{K}(\lambda))$ (red pluses) and method of Shen et al. (blue circles).

Zeros of analytic function $\det_1(\mathcal{I} + \mathcal{K}(\lambda))$ were computed via Lyness' method: Contour integrals + Newton.

CGLE: Behaviour near edge of $\sigma_{\text{ess}}(\mathcal{L})$

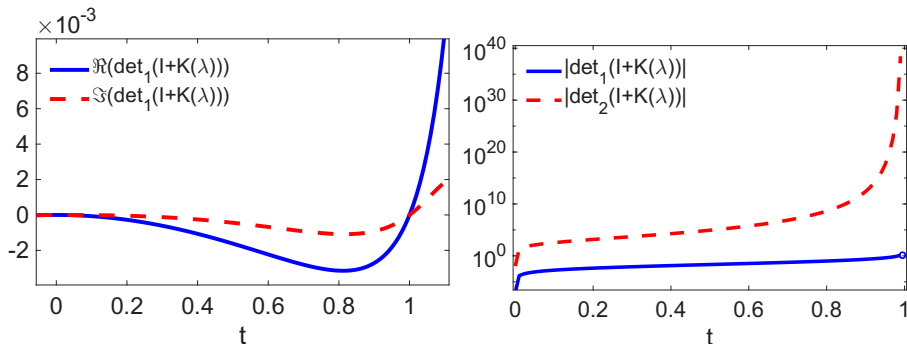


Figure: **Left:** Slice along line $\lambda(t) = \lambda_c t$, between eigenvalues at $\lambda = 0$ and $\lambda_c \approx -0.0033 + 0.0704i$. **Right:** Slice along line joining λ_c to λ_e .

$$\det_2(\mathcal{I} + \mathcal{K}(\lambda)) = e^{-\text{Tr}(\mathcal{K}(\lambda))} \det_1(\mathcal{I} + \mathcal{K}(\lambda)).$$

$$\text{and } \text{Tr}(\mathcal{K}(\lambda)) = \frac{a_+}{\sqrt{\lambda - \lambda_{e,+}}} + \frac{a_-}{\sqrt{\lambda - \lambda_{e,-}}} \rightarrow -\infty \text{ as } \lambda \rightarrow \lambda_{e,\pm}.$$

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