LAST NAME:	FIRST NAME:	CIRCLE:	Akbar 4pm	Coskunuzer 8:30am
GAUSS	CARL	Coskunuzer 10am	Zweck 1pm	

MATH 2415 [Spring 2023] Exam II

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points. **Your points for each problem will be recorded on the top of the second page.**

(1) [9 pts] Find an equation for the tangent plane to the surface $z = 2x^2 + 4y^2$ at the point $(1, 1, 6)$.

$$z = f(x,y) = 2x^2 + 4y^2$$

$$\frac{\partial f}{\partial x} = 4x = 4 \quad @ (1,1,6)$$

$$\frac{\partial f}{\partial y} = 8y = 8 \quad @ (1,1,6)$$

$$f(1,1) = 6 \qquad (x_0, y_0) = (1,1)$$

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$z = 6 + 4(x-1) + 8(y-1)$$

$$z = 4x + 8y - 6$$

1 /9	2 /12	3 /15	4 /15	5 /12	6 /12	T /75

(2) [12 pts]
(a) Let $(x, y) = \mathbf{r}(t) = (\sqrt{2}\cos t, 2\sqrt{2}\sin t)$ and let $z = f(x, y)$ be a function for which $f(1, 2) = 3$, $\frac{\partial f}{\partial x}(1, 2) = 4$ and $\frac{\partial f}{\partial y}(1, 2) = 5$. Let $g(t) = f(\mathbf{r}(t))$. Find $g'(\frac{\pi}{4})$.

$$g(t) = (f \circ \vec{r})(t)$$

$$g'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$g'(\pi/4) = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2) \right) \cdot \vec{r}\,'(\pi/4)$$

$$= (4, 5) \cdot (-1, 2)$$

$$\boxed{g'(\pi/4) = 6}$$

$$\vec{r}(\pi/4) = \left(\sqrt{2}\,\tfrac{1}{\sqrt{2}}, 2\sqrt{2}\,\tfrac{1}{\sqrt{2}}\right)$$

$$= (1, 2)$$

$$\vec{r}\,'(t) = (-\sqrt{2}\sin t, 2\sqrt{2}\cos t)$$

$$\vec{r}\,'(\pi/4) = (-1, 2)$$

(b) Show that the function $u(x, y) = x^3 \pm 3xy^2$ satisfies Laplace's equation, $u_{xx} + u_{yy} = 0$.

$$u_{x} = 3x^2 + 3y^2$$

$$u_{xx} = 6x$$

$$u_y = -6xy$$

$$u_{yy} = -6x$$

So $u_{xx} + u_{yy} = 6x - 6x = 0.$

(3) [15 pts] Let $f(x,y) = xe^y$.

(a) Find the directional derivative of $f(x,y) = xe^y$ at the point $(2,0)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(e^y, xe^y\right) = \left(e^0, 2e^0\right)$$

$$= (1,2) \text{ @ } (2,0)$$

$$\vec{u} = \frac{3\vec{i}+4\vec{j}}{|3\vec{i}+4\vec{j}|} = \frac{1}{5}(3,4)$$

$$(D_{\vec{u}} f)(2,0) = \nabla f(2,0)\cdot\vec{u} = \frac{(1,2)\cdot(3,4)}{5} = \boxed{\frac{11}{5}}$$

(b) In what direction does f have the maximum rate of change at the point $(2,0)$? What is this maximum rate of change?

- $$\vec{u} = \frac{\nabla f(2,0)}{|\nabla f(2,0)|} = \frac{(1,2)}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}}(1,2)$$
- MAX RofC $= |\nabla f(2,0)| = \sqrt{5}$

(c) In what directions is the rate of change of f at the point equal to zero at the point $(2,0)$?

FIND $\vec{u}$ WITH $|\vec{u}| = 1$ SO THAT

$$0 = (D_{\vec{u}} f)(2,0) = \nabla f(2,0)\cdot\vec{u}$$

$$\boxed{0 = (1,2)\cdot\vec{u}} \qquad \vec{u} = (a,b)$$

$$0 = (1,2)\cdot(a,b) = a + 2b$$

Set $\cancel{a}$ $b = 1$, $a = -2$ SO

$$\vec{u} = \frac{(-2,1)}{\sqrt{5}} \text{ OR } \frac{(2,-1)}{\sqrt{5}}$$

GIVES LENGTH OF $\vec{u}$ IS 1.

(4) [15 pts] Let S be the surface with parametrization

$$(x,y,z) = \mathbf{r}(u,v) = \left(\sqrt{1+v^2}\cos u\,,\ \sqrt{1+v^2}\sin u\,,\ \frac{v}{2}\right), \qquad \text{for } 0 \le u \le 2\pi \text{ and } -2 \le v \le 2.$$

(a) Show that S is part of a hyperboloid of one sheet. **Hint:** Find an equation of the form $F(x,y,z) = 0$ for this surface.

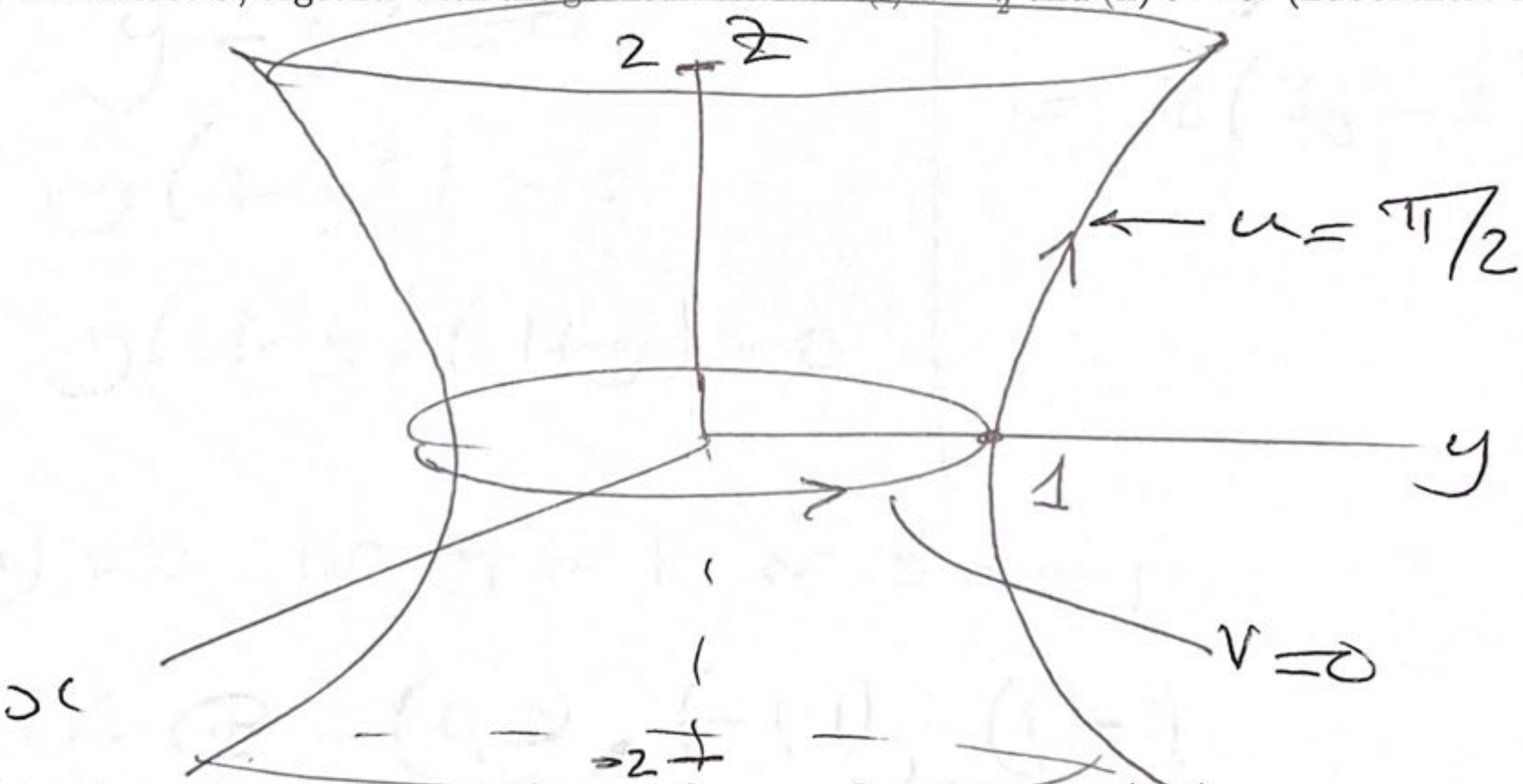
$x = \sqrt{1+v^2}\cos u$ $\qquad z = \frac{v}{2}$

$y = \sqrt{1+v^2}\sin u$ $\qquad v = 2z$

$x^2 + y^2 = (1+v^2)(\cos^2 u + \sin^2 u) = 1 + v^2 = 1 + 4z^2$

So $\boxed{x^2 + y^2 - 4z^2 = 1}$

(b) Sketch the surface S, together with the grid curves where (i) $u = \frac{\pi}{2}$ and (ii) $v = 0$. (Label these curves!)



(c) Calculate a tangent vector to the grid curve where $u = \frac{\pi}{2}$ at the point $\mathbf{r}(\frac{\pi}{2}, 0)$.

$$\vec{V} = \frac{\partial \vec{r}}{\partial v}(\pi/2, 0) = \left(\frac{1}{2}(1+v^2)^{-1/2}\, 2v \cos u,\ \frac{1}{2}(1+v^2)^{-1/2}\, 2v \sin u,\ \frac{1}{2}\right)$$

$$= \left(0, 0, \frac{1}{2}\right) \text{ @ } u = \pi/2,\ v = 0$$

(5) [12 pts] Find and classify all critical points of the function $f(x,y) = 2x^2 + y^4 + 4xy$.

$$\nabla f = (4x + 4y,\ 4y^3 + 4x) = (0, 0)$$

gives $y = -x$ ①

$x = -y^3$ ②

Plug ① into ② to get

$$y - y^3 = 0$$
$$y(1-y^2) = 0$$
$$y(1-y)(1+y) = 0$$

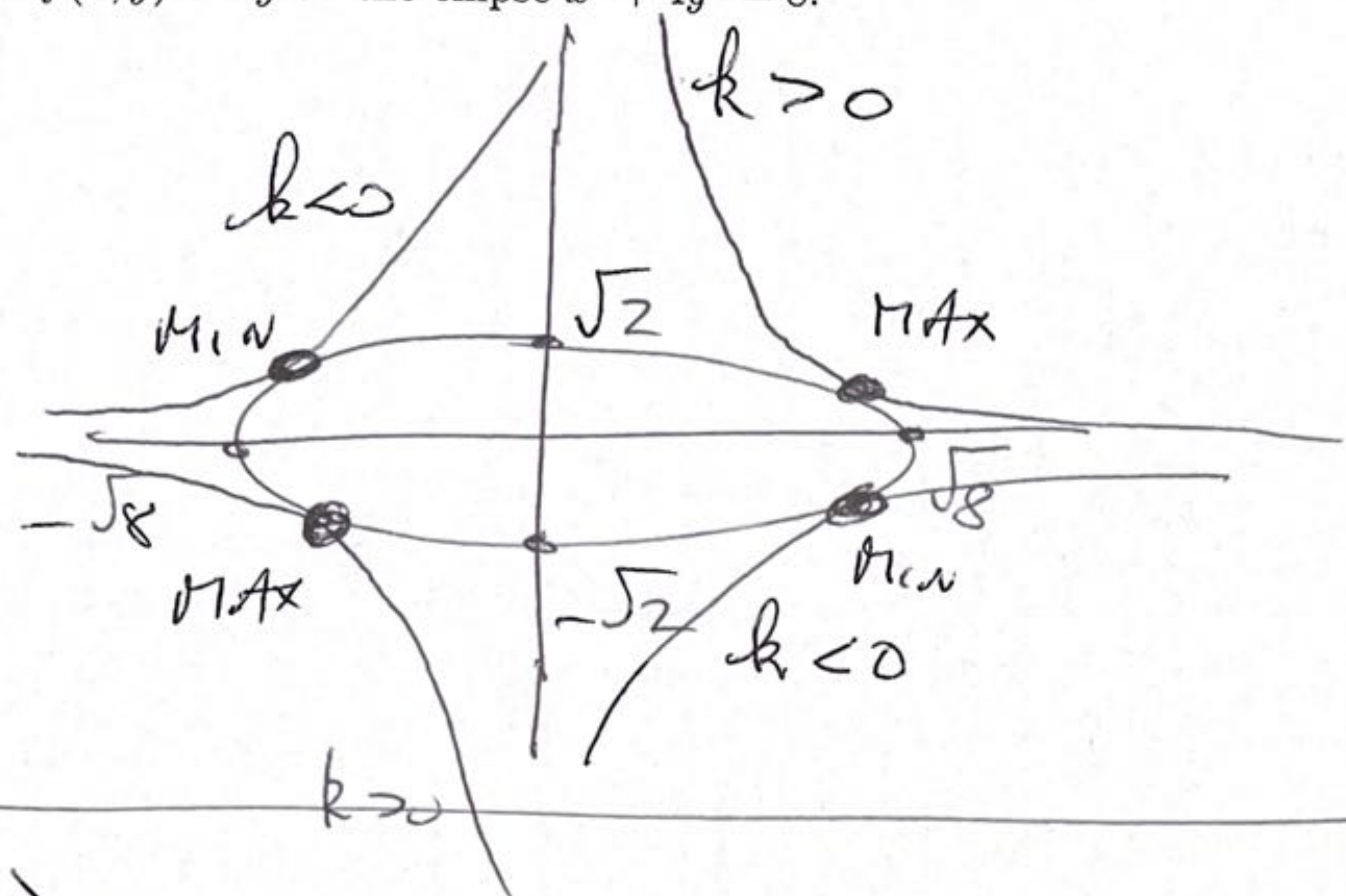
$$D = \mathrm{DET}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$
$$= \begin{vmatrix} 4 & 4 \\ 4 & 12y^2 \end{vmatrix}$$
$$= 48y^2 - 16$$
$$= 16(3y^2 - 1)$$

$y = 0$ or $y = 1$ or $y = -1$

So 3 CPTS @ $(0,0)$, $(-1,1)$, $(1,-1)$

CPT	D	f_{xx}	CLASSIFICATION
$(0,0)$	$-16<0$		SADDLE
$(-1,1)$	$32>0$	$4>0$	LOCAL MIN
$(1,-1)$	$32>0$	$4>0$	LOCAL MIN

(6) [12 pts] Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function $f(x,y) = xy$ on the ellipse $x^2 + 4y^2 = 8$.



LEVEL CURVES

$f(x,y) = k$

4 PTS OF COMMON TANGENCY.

SUGGESTS 4 CPTS

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$: $y = \lambda 2x$ (1)

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$: $x = \cancel{2}8\lambda y$ (2)

$g = c$: $x^2 + 4y^2 = 8$ (3)

SUB (1) INTO (2) : $x = 8\lambda^2 2x$

$x(1 - 16\lambda^2) = 0$

$x = 0$ OR $\lambda = \pm\frac{1}{4}$

$\boxed{x=0} \Rightarrow y = 0$ BY (1) $\Rightarrow 0 = 8$ by (3) NO SOLNS

$\boxed{\lambda = \frac{+1}{4}}$ $y = \frac{1}{2}x$ by (1) So by (3) $x^2 + x^2 = 8$

$x = \pm 2$

$y = \pm 1$

$(x,y,\lambda) =$

$(2, 1, \frac{1}{4})$, $f(2,1) = 2$

$(-2, -1, 1/4)$ $f(-2,-1) = 2$

$\lambda = -1/4$ $y = -\frac{1}{2}x$, by (1) So by (3) $x = \pm 2$, $y = \mp 1$

$(x\,y\,\lambda) = (-2, +1, -1/4)$ $f = -2$ OR $(2, -1, -1/4)$ $f = -2$

(x,y)	λ	f	
$(2,1)$	$\frac{1}{4}$	2	MAX
$(-2,-1)$	$\frac{1}{4}$	2	MAX
$(2,-1)$	$-\frac{1}{4}$	-2	MIN
$(-2,1)$	$-\frac{1}{4}$	-2	MIN