LAST NAME:	FIRST NAME:	CIRCLE: Akbar 4pm	Coskunuzer 8:30am
MAXWELL	JAMES	Coskunuzer 10am	Zweck 1pm

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MATH 2415 [Spring 2023] Exam I

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points. **Your points for each problem will be recorded on the top of the second page.**

(1) [12 pts] Let $\mathbf{u} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - 2\mathbf{k}$.

(a) Find the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

$$\text{PROJ}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\vec{u} = \frac{(3,-1,0)\cdot(1,5,-2)}{\left(\sqrt{1^2+5^2+(-2)^2}\right)^2}(1,5,-2)$$

$$= \frac{-2}{30}(1,5,-2)$$

$$= \frac{1}{15}(-1,-5,2)$$

(b) Find the volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$\text{VOL} = |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

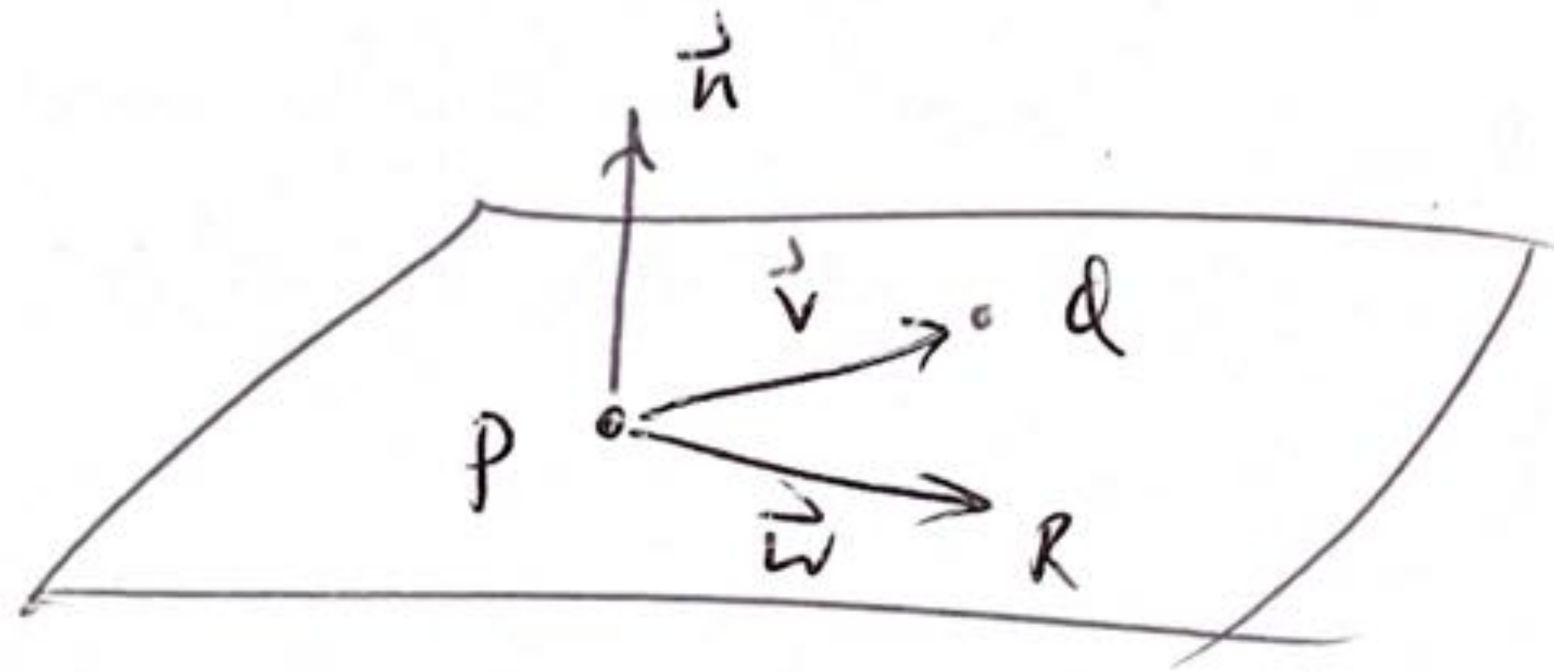
$$= \left| \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -2 \end{vmatrix} \right|$$

$$= |1(2-0) - 5(-6-0) - 2(27+5)|$$

$$= |2+30-64| = |-32| = \boxed{32}$$

1 /12	2 /12	3 /13	4 /12	5 /14	6 /12	T /75

(2) [12 pts] Consider the three points $P = (1, 0, 1)$, $Q = (-2, 1, 3)$, and $R = (4, 2, 5)$.
(a) Find a unit vector that is perpendicular to the plane containing these three points.



$$\vec{v} = \overrightarrow{PQ} = (-2, 1, 3) - (1, 0, 1) = (-3, 1, 2)$$

$$\vec{w} = \overrightarrow{PR} = (4, 2, 5) - (1, 0, 1) = (3, 2, 4)$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= 0\vec{i} - (-12 - 6)\vec{j} + (-6-3)\vec{k}$$

$$= 18\vec{j} - 9\vec{k}$$

$$= 9(2\vec{j} - \vec{k})$$

$$\vec{n} = \frac{2\vec{j} - \vec{k}}{|2\vec{j} - \vec{k}|} = \frac{1}{\sqrt{5}}(0, 2, -1)$$

(b) Find the area of the triangle PQR.

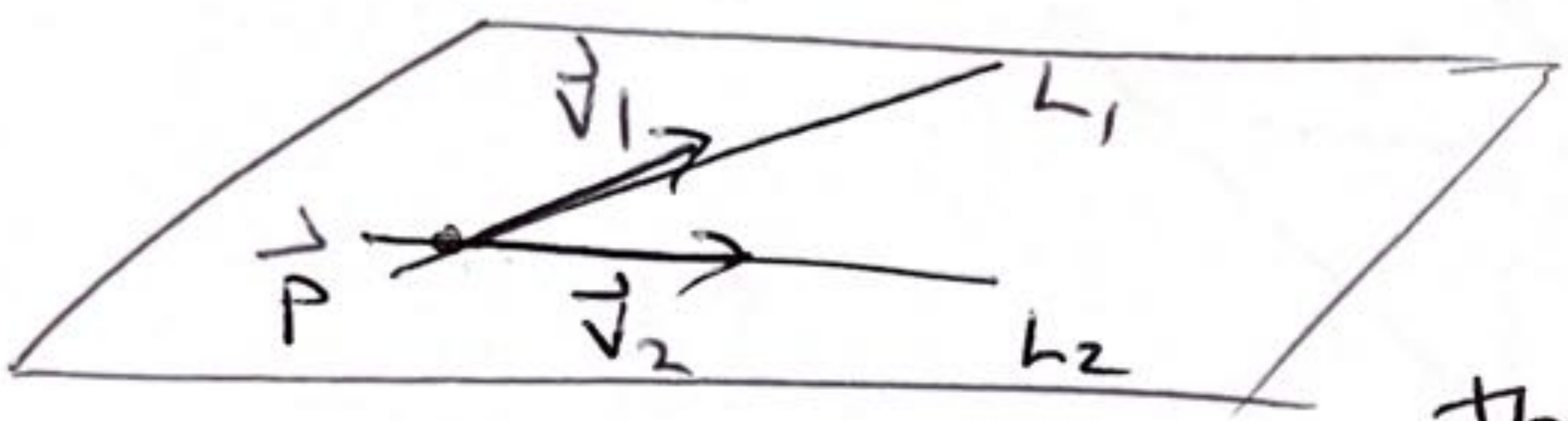
$$A = \frac{1}{2}|\vec{v} \times \vec{w}| = \frac{1}{2}|18\vec{j} - 9\vec{k}|$$

$$= \frac{9}{2}|2\vec{j} - \vec{k}|$$

$$= \frac{9}{2}\sqrt{5}$$

(3) [13 pts]

(a) Suppose that two lines, L_1 and L_2, intersect in a point. Draw a schematic diagram that illustrates why there is exactly one plane $\mathcal{P}$ than contains both L_1 and L_2. Write a brief explanation of your diagram.



SINCE $L_2 \neq L_1$, the vectors $\vec{v}_1, \vec{v}_2$ are not parallel. So there is exactly one plane thru point $\vec{p}$ that contains vectors $\vec{v}_1, \vec{v}_2$.

This plane contains both lines.

For the rest of the problem suppose that L_1 is the line parametrized by $\mathbf{r}_1(t) = (2t+1, 3t-2, t+1)$ and L_2 is the line parametrized by $\mathbf{r}_2(s) = (s+1, 2s-3, 3s-4)$.

(b) Show that the point $\mathbf{p} = (3, 1, 2)$ lies on both lines.

$\vec{r}_1(1) = (3, 1, 2) = \vec{p}$ So $\vec{p}$ lies on both lines

$\vec{r}_2(2) = (3, 1, 2) = \vec{p}$

(c) Find an equation of the form $Ax + By + Cz = D$ for the plane $\mathcal{P}$ than contains both L_1 and L_2.

$\vec{v}_1 = (2, 3, 1)$, $\vec{v}_2 = (1, 2, 3)$, $\vec{p} = (3, 1, 2)$.

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 7\vec{i} - 5\vec{j} + 1\vec{k}$$

So $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$ gives

$(x-3, y-1, z-2) \cdot (7, -5, 1) = 0$

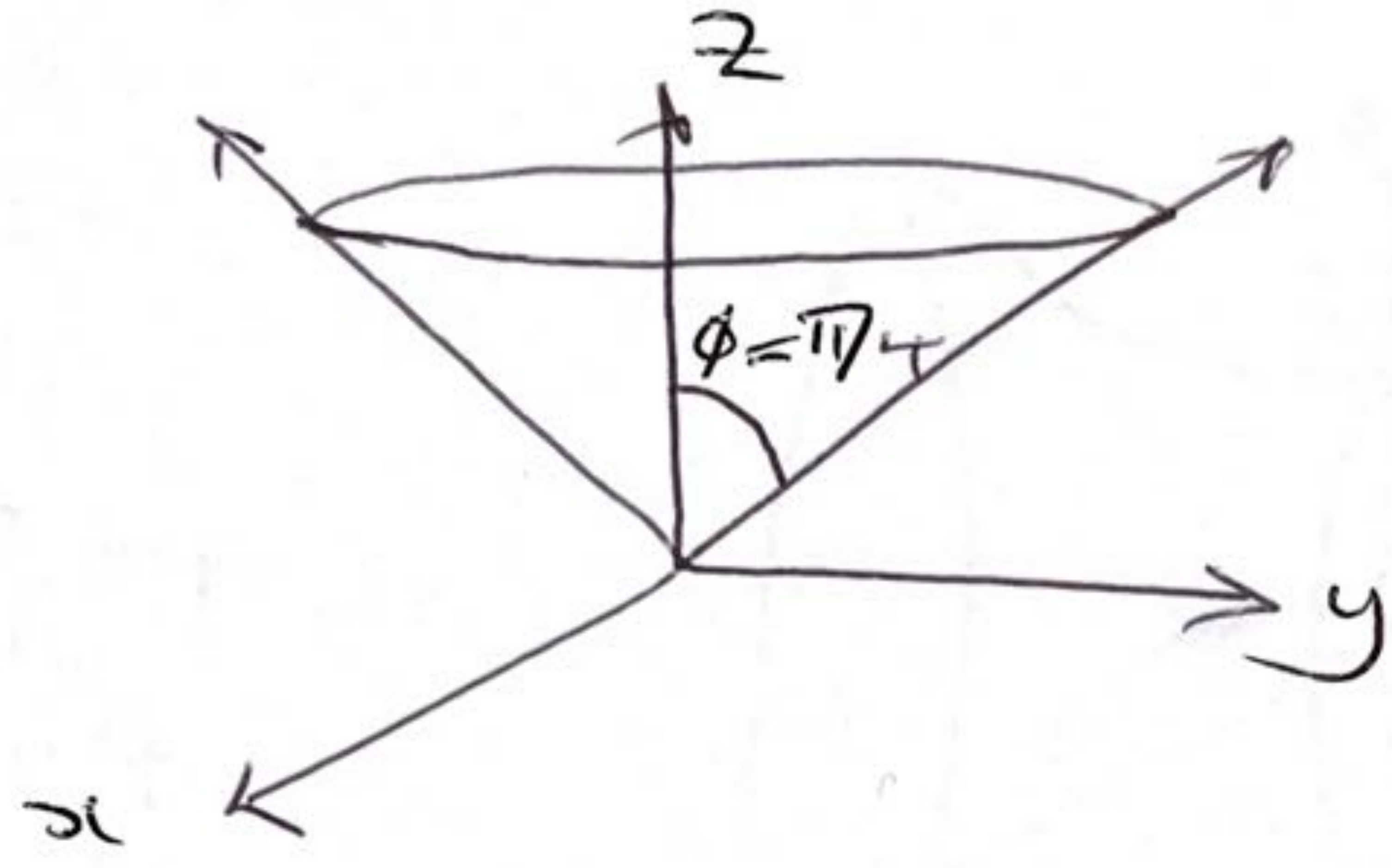
$7(x-3) - 5(y-1) + 1(z-2) = 0$

$7x - 5y + z = 21 - 5 + 2 = 18$

$\boxed{7x - 5y + z = 18}$

(4) [12 pts]

(a) Sketch the surface whose equation in spherical coordinates, (ρ, θ, ϕ), is $\phi = \pi/4$.



(b) Convert the equation $\phi = \pi/4$ to rectangular coordinates.

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

So $x^2 + y^2 = \rho^2 \sin^2\phi$

$$\frac{x^2+y^2}{z^2} = \tan^2\phi = 1$$

So $z^2 = x^2 + y^2$.

NOW $z = \rho \cos \pi/4 > 0$

So $\boxed{z = \sqrt{x^2+y^2}}$

(c) Convert the equation $\phi = \pi/4$ to cylindrical coordinates.

$z^2 = x^2 + y^2$

$z^2 = r^2$

$z = \pm r$

BUT $z > 0$ So $\boxed{z = r}$

(5) [14 pts]

(a) Sketch the surface $x^2 + z^2 - 2y^2 = 4$. **Hint:** You may find it helpful to use cylindrical coordinates (r, θ, y) (instead of the usual (r, θ, z)).

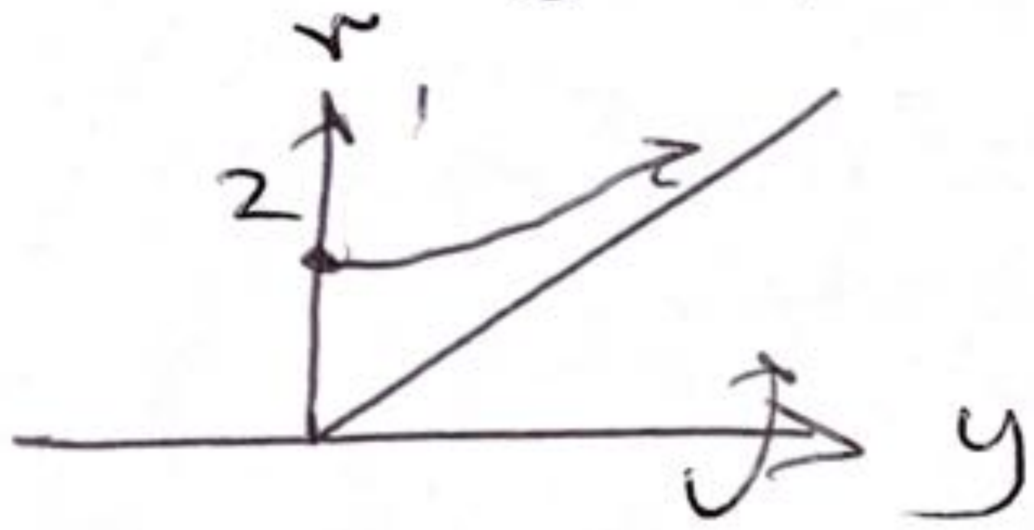
$x = r\cos\theta$

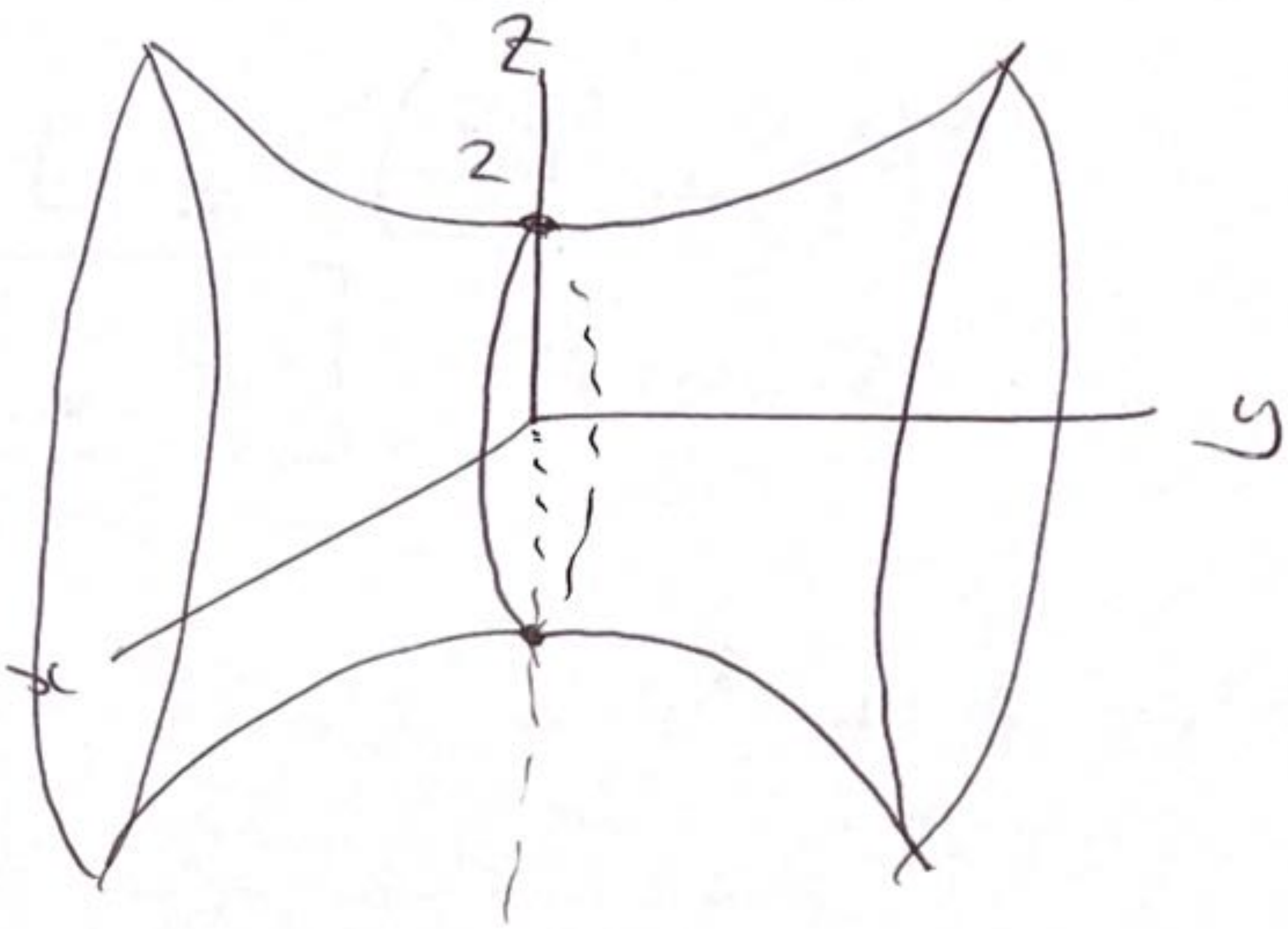
$z = r\sin\theta$

$y = y.$

$x^2 + z^2 - 2y^2 = 4$

is $\quad r^2 - 2y^2 = 4$

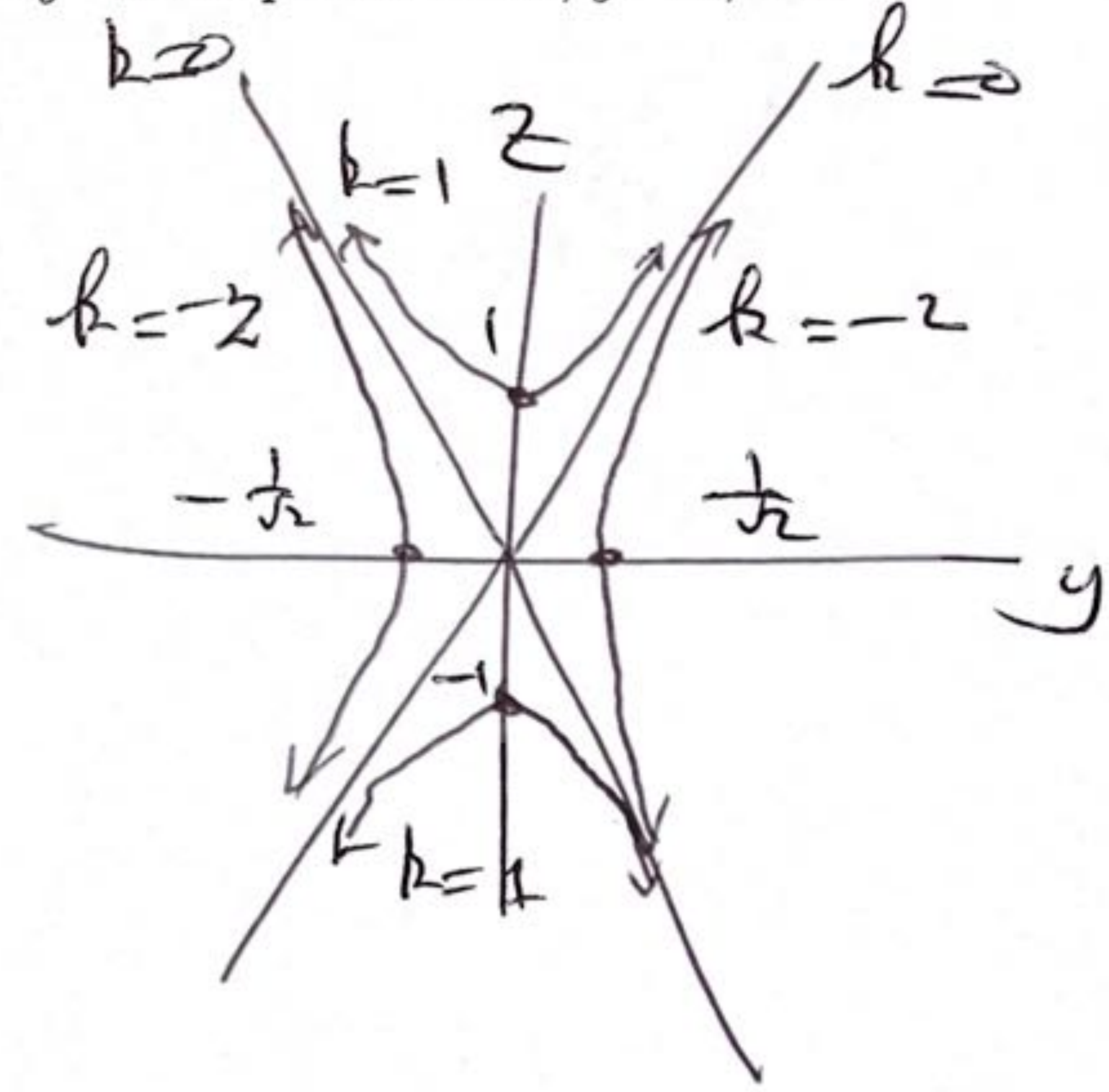




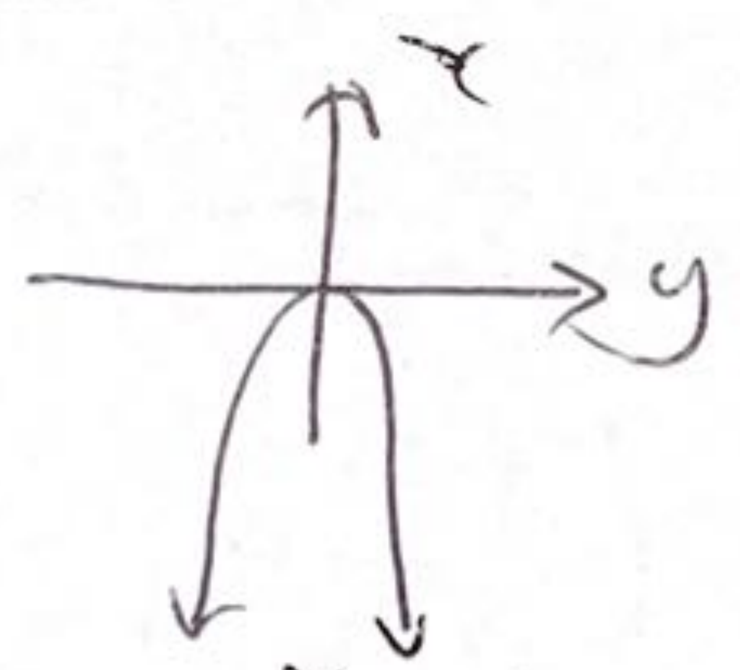
Intercept: $y = 0,\ r = 2$

Asymptote $\quad r = \sqrt{2}\,y$

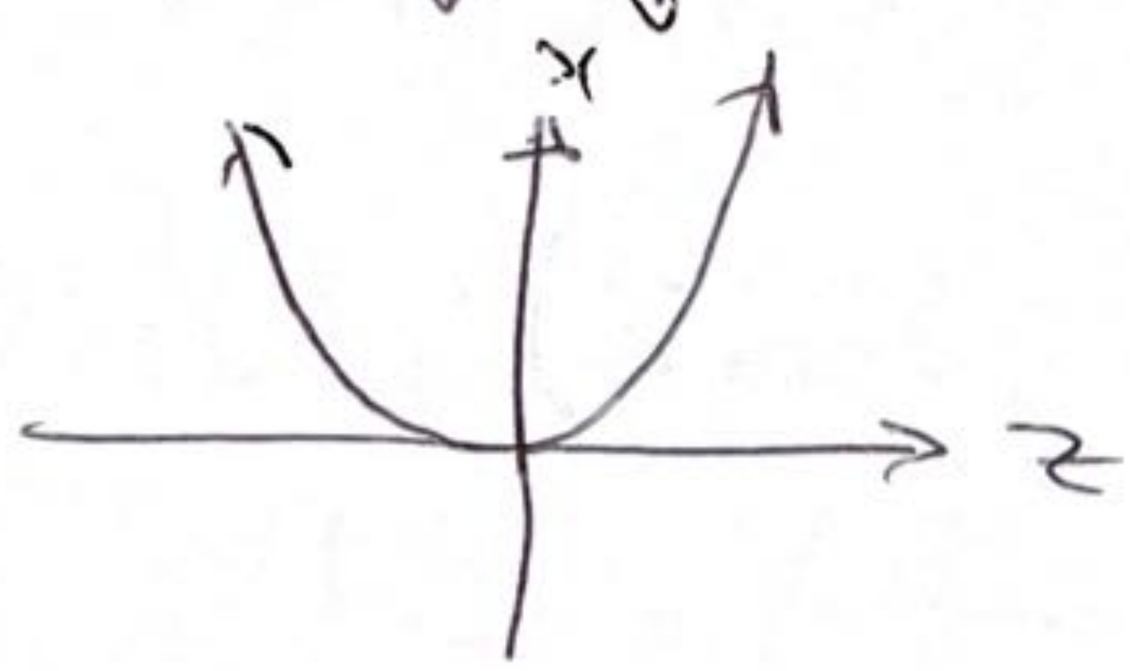
(b) Make *labelled* sketches of the traces (slices) of the surface $x = z^2 - 4y^2$ in the planes $z = 0$, $y = 0$, and $x = k$, for $k = 0, 1, -2$. (You do not need to sketch the surface itself.)



$\underline{z=0}$ $\quad x = -4y^2$



$\underline{y=0}$ $\quad x = z^2$



$x = k: \quad z^2 - 4y^2 = k$

$\underline{k=0}$ $\quad z = \pm 2y$

$\underline{k=1}$ $\quad z^2 - 4y^2 = 1$

Intercept; $y = 0,\ z = \pm 1$

$\underline{k=-2}$

$z^2 - 4y^2 = -2$

Intercept: $\cancel{z} = 0$

$y = \pm \frac{1}{\sqrt{2}}$

(6) [12 pts] Let C be the curve parametrized by $\mathbf{r}(t) = (3\sin t, 4, 3\cos t)$.
(a) Show that C lies on a cylinder and on a sphere.

$x = 3\sin t$
$y = 4$
$z = 3\cos t$

CYLINDER

$$1 = \cos^2 t + \sin^2 t = \left(\frac{z}{3}\right)^2 + \left(\frac{x}{3}\right)^2$$

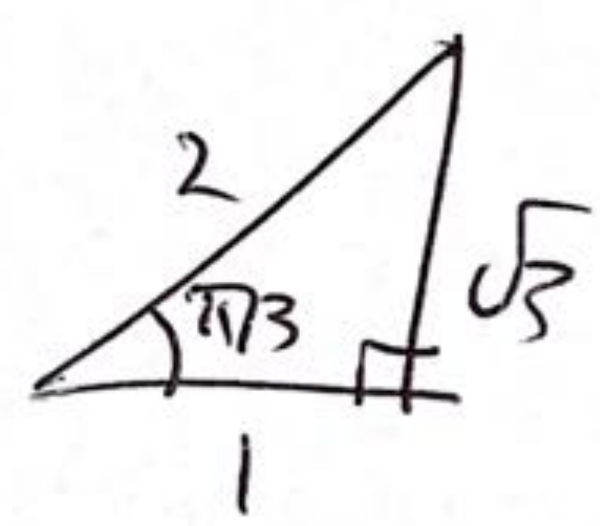
So $\boxed{x^2 + z^2 = 9.}$ CYLINDER

SPHERE

$$x^2 + y^2 + z^2 = 9\sin^2 t + 4^2 + 9\cos^2 t$$
$$= 3^2 + 4^2 = 5^2$$

$x^2 + y^2 + z^2 = 5^2$ ⤳ SPHERE

(b) Find a parametrization for the tangent line to C at the point where $t = \pi/3$.



$$\vec{p} = \vec{r}(\pi/3) = (3\sin \pi/3, 4, 3\cos \pi/3)$$
$$= \left(\frac{3\sqrt{3}}{2}, 4, \frac{3}{2}\right)$$

$\vec{v} = \vec{r}\,'(\pi/3)$, $\quad \vec{r}\,'(t) = (3\cos t, 0, -3\sin t)$

$$\vec{v} = \vec{r}\,'(\pi/3) = \left(\frac{3}{2}, 0, -\frac{3\sqrt{3}}{2}\right)$$

$$\vec{\ell}(s) = \vec{p} + (s - \pi/3)\vec{v}$$
$$= \left(\frac{3\sqrt{3}}{2}, 4, \frac{3}{2}\right) + (s - \pi/3)\left(\frac{3}{2}, 0, \frac{-3\sqrt{3}}{2}\right)$$