LAST NAME:	FIRST NAME:	CIRCLE:	Coskunuzer 8:30am	Ahsan 1pm
LEIBNIZ	GOTTFRIED		Ahsan 2:30pm	Zweck 4pm

MATH 2415 [Fall 2024] Exam II

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. This 75 minute exam is worth 75 points. **Points will be recorded on the top of the second page.**

(1) [12 pts] Suppose that $z = f(x, y) = \cos(2x + 3y)$ where $x = x(t)$ and $y = y(t)$. If $x(0) = -\pi/4$, $y(0) = \pi/3$, $x'(0) = 5$, and $y'(0) = 4$, find $\frac{dz}{dt}$ at $t = 0$.

$\vec{r}(t) = (x(t), y(t))$

$\vec{r}(0) = (-\pi/4, \pi/3)$

$\vec{r}\,'(0) = (5, 4)$

$\nabla f = (-2\sin(2x+3y), -3\sin(2x+3y))$

$\nabla f(\vec{r}(0)) = (-2\sin(-\pi/2+\pi), -3\sin(-\pi/2+\pi))$

$= (-2\sin\pi/2, -3\sin\pi/2)$

$= (-2, -3)$

$z(t) = (f\circ\vec{r})(t)$

$\frac{dz}{dt} = (f\circ\vec{r})'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$\frac{dz}{dt}(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$

$= (-2, -3) \cdot (5, 4)$

$= -22$

1	/12	2	/13	3	/13	4	/12	5	/13	6	/12	T	/75

(2) [13 pts] Let $f(x, y) = x^2 + \cos y + 2ye^x$ and let $\mathbf{x}_0 = (0, \frac{\pi}{2})$.

(a) Find the gradient of f at $\mathbf{x}_0$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2x + 2ye^x, -\sin y + 2e^x\right)$$

$$\nabla f(0, \pi/2) = (\pi, -1 + 2) = \cancel{\pi 2} \; (\pi, 1)$$

(b) Find the directional derivative of f at $\mathbf{x}_0$ in the direction of the vector $\mathbf{v} = (4, 3)$.

$$D_{\vec{u}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(4,3)$$

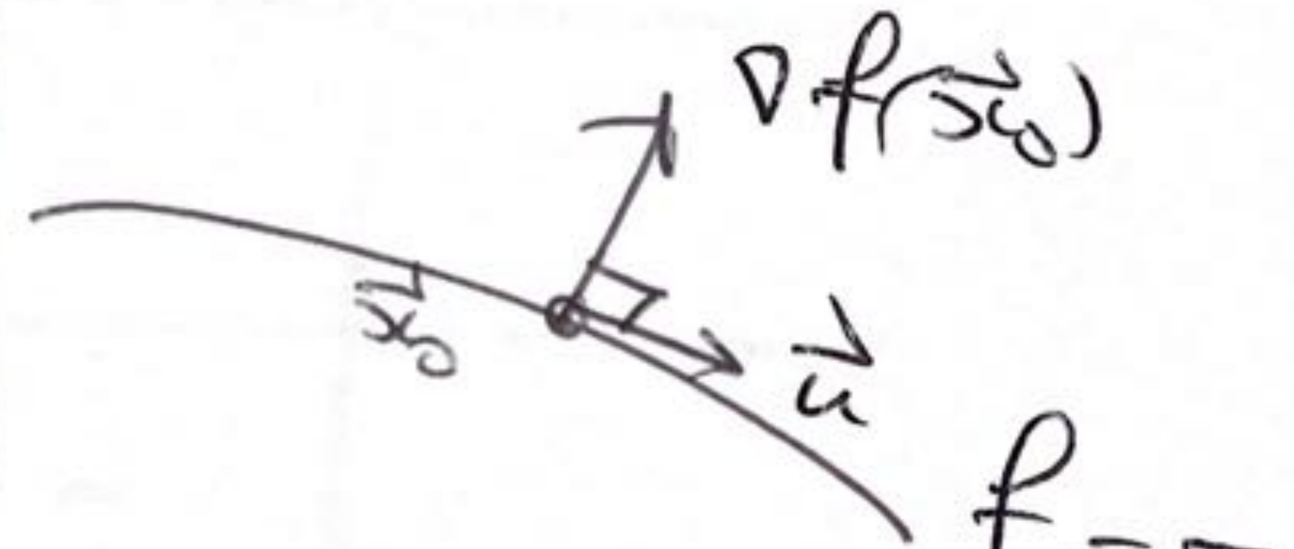
$$= (\pi, 1) \cdot \frac{1}{5}(4,3) = \frac{4\pi + 3}{5}$$

(c) Find the minimum (i.e. most negative) rate of change of f at $\mathbf{x}_0$ and the direction in which it occurs.

MIN R of C $= -|\nabla f(\vec{x}_0)| = -\sqrt{\pi^2 + 1}$

DIR^n ~~is~~ $\vec{u} = \dfrac{-\nabla f(\vec{x}_0)}{|\nabla f(\vec{x}_0)|} = \dfrac{-(\pi, 1)}{\sqrt{\pi^2 + 1}}$

(d) Which of the following vectors (if any) are tangent to the curve $f(x, y) = \pi$ at the point $\mathbf{x}_0$: $\mathbf{u} = (1, \pi)$, $\mathbf{v} = (1, -\pi)$, and $\mathbf{w} = (2\pi, 2)$. Explain!



Need $\vec{u} \cdot \nabla f(\vec{x}_0) = 0$

$$\vec{u} \cdot \nabla f(\vec{x}_0) = (1, \pi) \cdot (\pi, 1) = 2\pi \neq 0$$

$$\vec{v} \cdot \nabla f(\vec{x}_0) = (1, -\pi) \cdot (\pi, 1) = -\pi + \pi = 0$$

$$\vec{w} \cdot \nabla f(\vec{x}_0) = (2\pi, 2) \cdot (\pi, 1) = 2\pi^2 + 2 \neq 0$$

So $\vec{v}$ IS TANGENT. $\vec{u}, \vec{w}$ are NOT

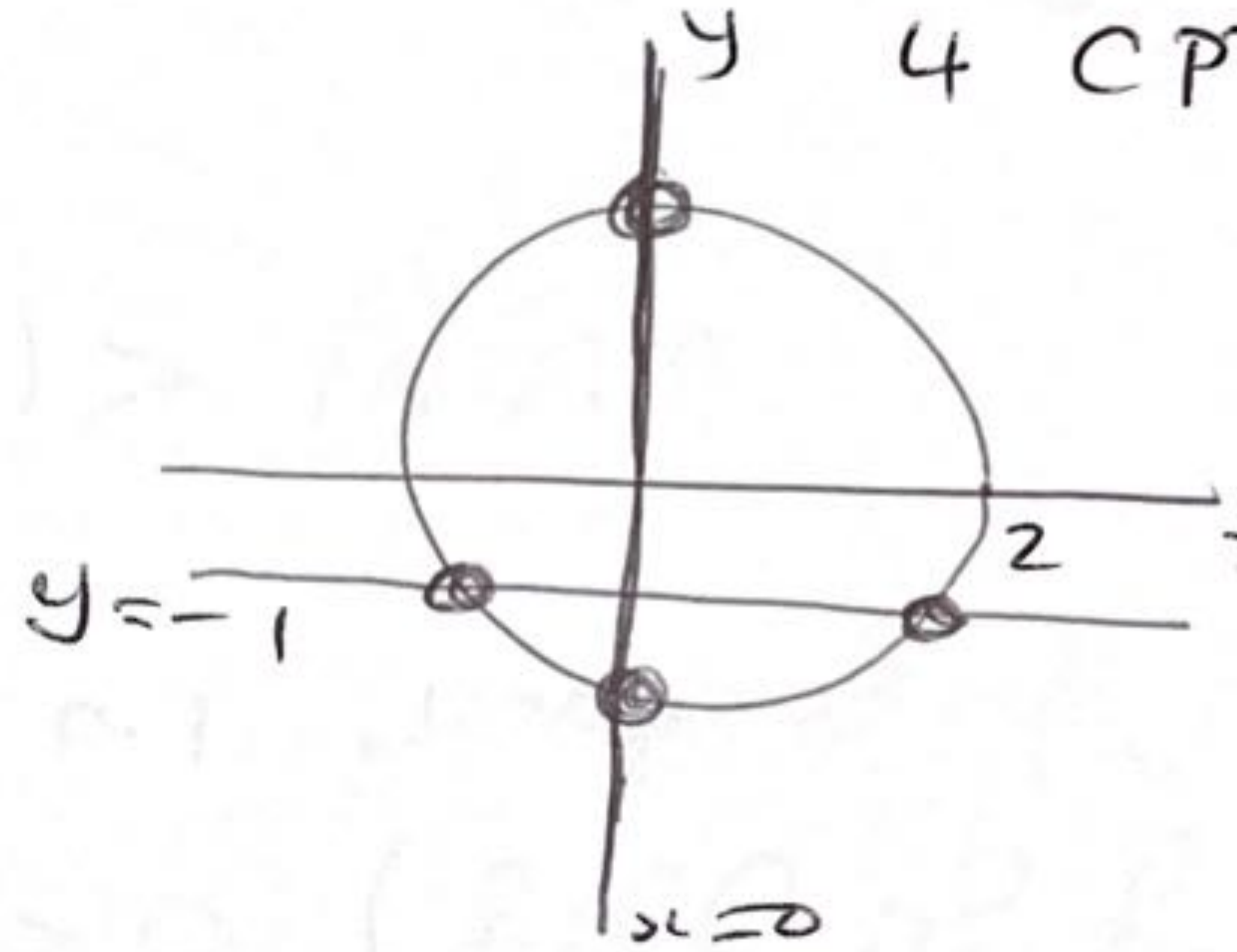
(3) [13 pts] Find and classify all critical points of the function $f(x,y) = y^3 - 12y + 3x^2y + 3x^2$.

$0 = f_x = 6xy + 6x \implies x(y+1) = 0 \quad (1)$

$0 = f_y = 3y^2 - 12 + 3x^2 \implies x^2 + y^2 = 4 \quad (2)$

(1) Says $x = 0$ or $y = -1$

4 CPTS



$\boxed{\text{CASE } x = 0}$

By (2) $y^2 = 4$

$y = \pm 2$

$\boxed{(x,y) = (0, \pm 2)}$

$\boxed{\text{CASE } y = -1}$ $x^2 = 3$

$x = \pm\sqrt{3}$

$(x,y) = (\pm\sqrt{3}, -1)$

$D = \mathrm{DET}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$= \mathrm{DET}\begin{bmatrix} 6y+6 & 6x \\ 6x & 6y \end{bmatrix}$

$= 36[y(y+1) - x^2]$

$f_{xx} = 6y + 6$

CPT	D	f_{xx}	TYPE
$(0,2)$	+	+	LOCAL MIN
$(0,-2)$	+	−	LOCAL MAX
$(\sqrt{3},-1)$	−		SADDLE POINT
$(-\sqrt{3},-1)$	−		SADDLE POINT

(4) [12 pts]

(a) Suppose $f_{xx}(0,0) = -1$, $f_{yy}(0,0) = 2$ and $f_{xy}(0,0) = 3$. Which would you expect to be larger (more positive) and why:

(i) $f_y(0,0)$ or $f_y(0,0.1)$?

As y increases from 0 to 0.1, EXPECT f_y to increase as $f_{yy}(0,0) > 0$ (Rate of Change of f_y wrt y is positive).

So expect $f_y(0, 0.1) > f_y(0,0)$

(ii) $f_x(0,0)$ or $f_x(0,0.1)$?

As y increases from 0 to 0.1 expect f_x to increase as $f_{xy}(0,0) > 0$ (R of C of f_x wrty is positive)

So expect $f_x(0, 0.1) > f_x(0,0)$

(b) Is there a function $z = f(x,y)$ so that $\frac{\partial f}{\partial x} = 2x\cos y$ and $\frac{\partial f}{\partial y} = x^2 \sin y$? Explain!

NO If there was we would need $f_{xy} = f_{yx}$

But $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(2x\cos y) = -2x\sin y$

and $f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(x^2 \sin y) = 2x\sin y$

So $f_{xy} \neq f_{yx}$

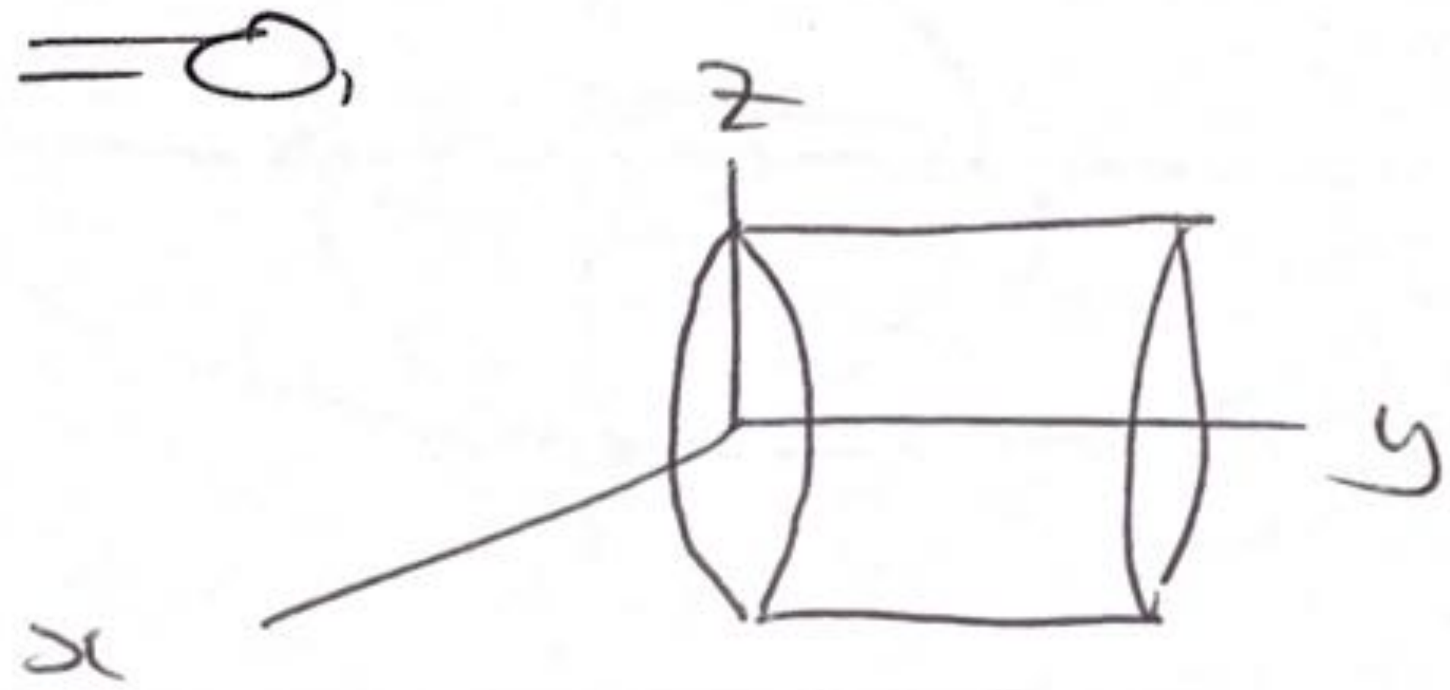
(5) [13 pts] Consider the parametrized surface $(x, y, z) = \mathbf{r}(u, v) = (\sqrt{2}\cos u, v, \sqrt{2}\sin \cancel{v}\, u)$.

(a) Find an equation of the form $F(x, y, z) = 0$ for this surface. ~~SKETCH SURFACE~~

$$x^2 + z^2 = (\sqrt{2}\cos u)^2 + (\sqrt{2}\sin u)^2 = 2$$

$$F(x,y,z) = x^2 + z^2 - 2 = 0,$$

CYLINDER



(b) Find an equation of the tanget plane to the surface at the point $\mathbf{r}(\frac{\pi}{4}, 1)$. Write your answer in the form $ax + by + cz + d = 0$.

$$\frac{\partial \vec{r}}{\partial u} = (-\sqrt{2}\sin u, 0, \sqrt{2}\cos u)$$

$$\frac{\partial \vec{r}}{\partial u}(\pi/4, 1) = (-1, 0, 1)$$

$$\frac{\partial \vec{r}}{\partial v} = (0, 1, 0)$$

$$\vec{p} = \vec{r}(\pi/4, 1) = (1, 1, 1)$$

$$\vec{n} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1, 0, -1)$$

So $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$ gives

$$-1(x-1) + 0(y-1) + -1(z-1) = 0$$

$$-x + 1 - z + 1 = 0$$

$$\boxed{x + z - 2 = 0}$$

(6) [12 pts] Use the Method of Lagrange Multipliers to find the absolute maximum and absolute minimum of the function $f(x, y) = 2x^2 + 3y^2$ subject to the constraint $x^2 + y^2 = 1$.

Level Curves of f:

$$2x^2 + 3y^2 = k$$

are Ellipses.

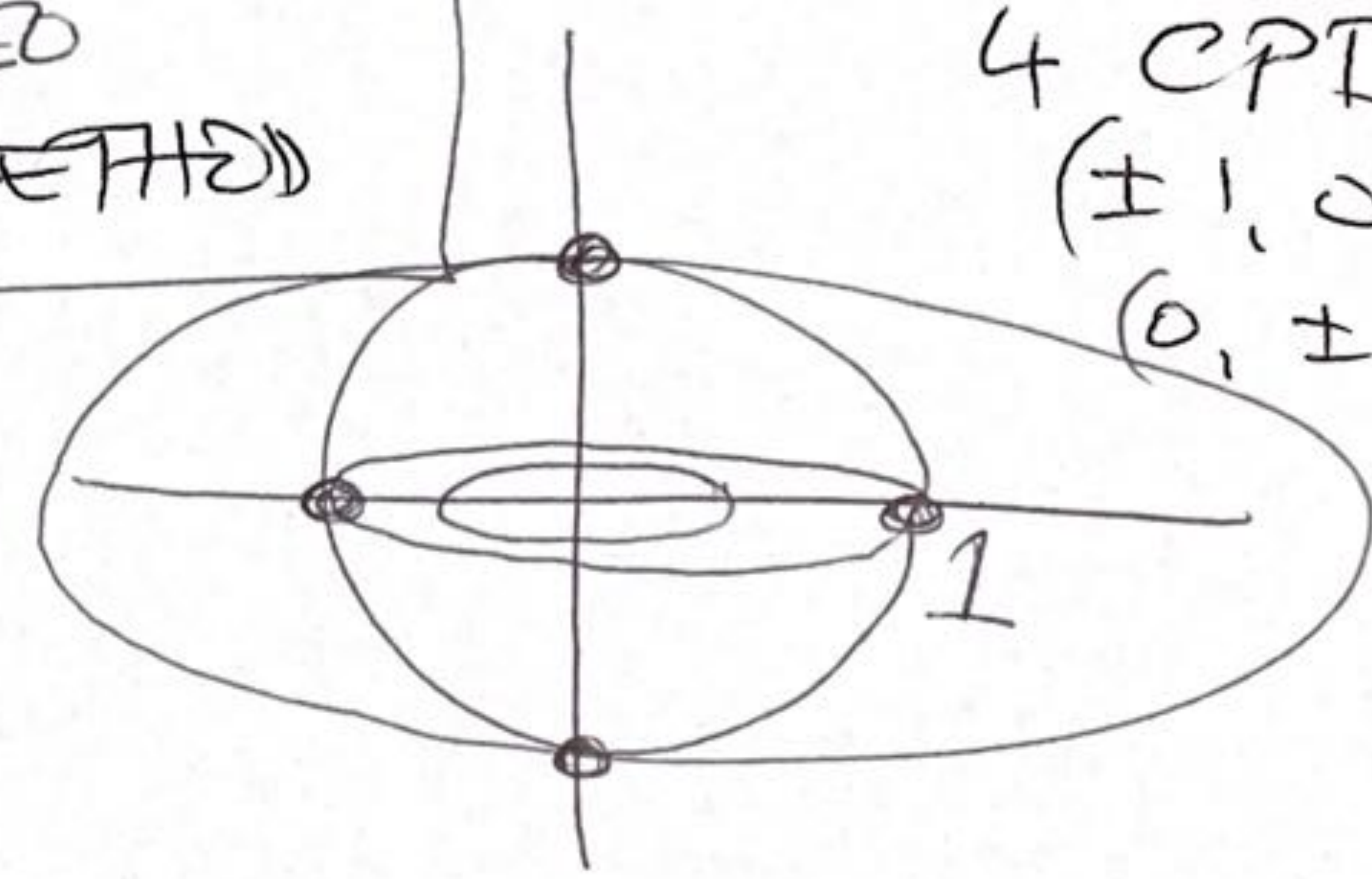
SHORTER WAY

GEO METHOD

4 CPTS

$(\pm 1, 0)$

$(0, \pm 1)$



ALG METHOD

$f_x = \lambda g_x$: $4x = 2\lambda x$ ①

$f_y = \lambda g_y$: $6y = 2\lambda y$ ②

$g = c$ $x^2 + y^2 = 1$ ③

By ① $x(2 - \lambda) = 0$

So $x = 0$ or $\lambda = 2$

$\boxed{x = 0}$ By ③ $y = \pm 1$

By ② $\pm 6 = \pm 2\lambda$

$\lambda = 3$

$(x, y, \lambda) = (0, \pm 1, 3)$

$\boxed{\lambda = 2}$ By ② $6y = 4y$

$y = 0$

So by ③ $x = \pm 1$

$(x, y, \lambda) = (\pm 1, 0, 2)$

(x, y)	$f(x, y)$	
$(1, 0)$	2	ABS MIN
$(-1, 0)$	2	ABS MIN
$(0, +1)$	3	ABS MAX
$(0, -1)$	3	ABS MAX