LAST NAME:	FIRST NAME:	CIRCLE:	Coskunuzer 8:30am	Ahsan 1pm
GREEN	GEORGE		Ahsan 2:30pm	Zweck 4pm

MATH 2415 [Fall 2024] Exam I

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. This 75 minute exam is worth 75 points. **Points will be recorded on the top of the second page.**

(1) [12 pts] Let $\mathbf{u} = \langle 2, 1, -3 \rangle$ and $\mathbf{v} = \langle -1, 2, 1 \rangle$.
(a) Find the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\mathrm{COMP}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{(2,1,-3) \cdot (-1,2,1)}{|(-1,2,1)|}$$

$$= \frac{-2+2-3}{\sqrt{1^2+2^2+1^2}} = -3/\sqrt{6}$$

(b) Find the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

$$\mathrm{PROJ}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2} \vec{u} = \frac{-3}{\left(\sqrt{2^2+1^2+3^2}\right)^2} (2,1,-3)$$

$$= \frac{-3}{14} (2,1,-3)$$

(c) Find a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

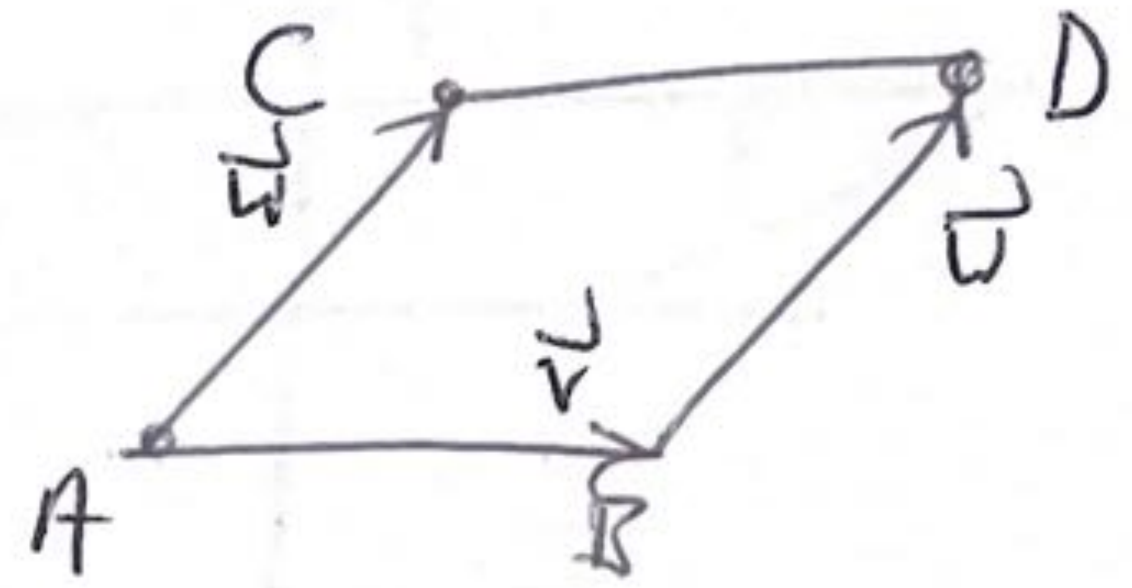
$$\vec{n} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} \qquad \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{vmatrix} = (7,1,5)$$

$$\text{So } \vec{n} = \frac{1}{\sqrt{75}} (7,1,5)$$

1 /12	2 /13	3 /13	4 /12	5 /13	6 /12	T /75

(2) [13 pts] Consider the points $A = (3, -1, 1)$, $B = (4, 2, -1)$ and $C = (2, 3, -3)$. Let AB and AC are two adjacent sides of a parallelogram $ABCD$.

(a) Find the coordinates of the point D.

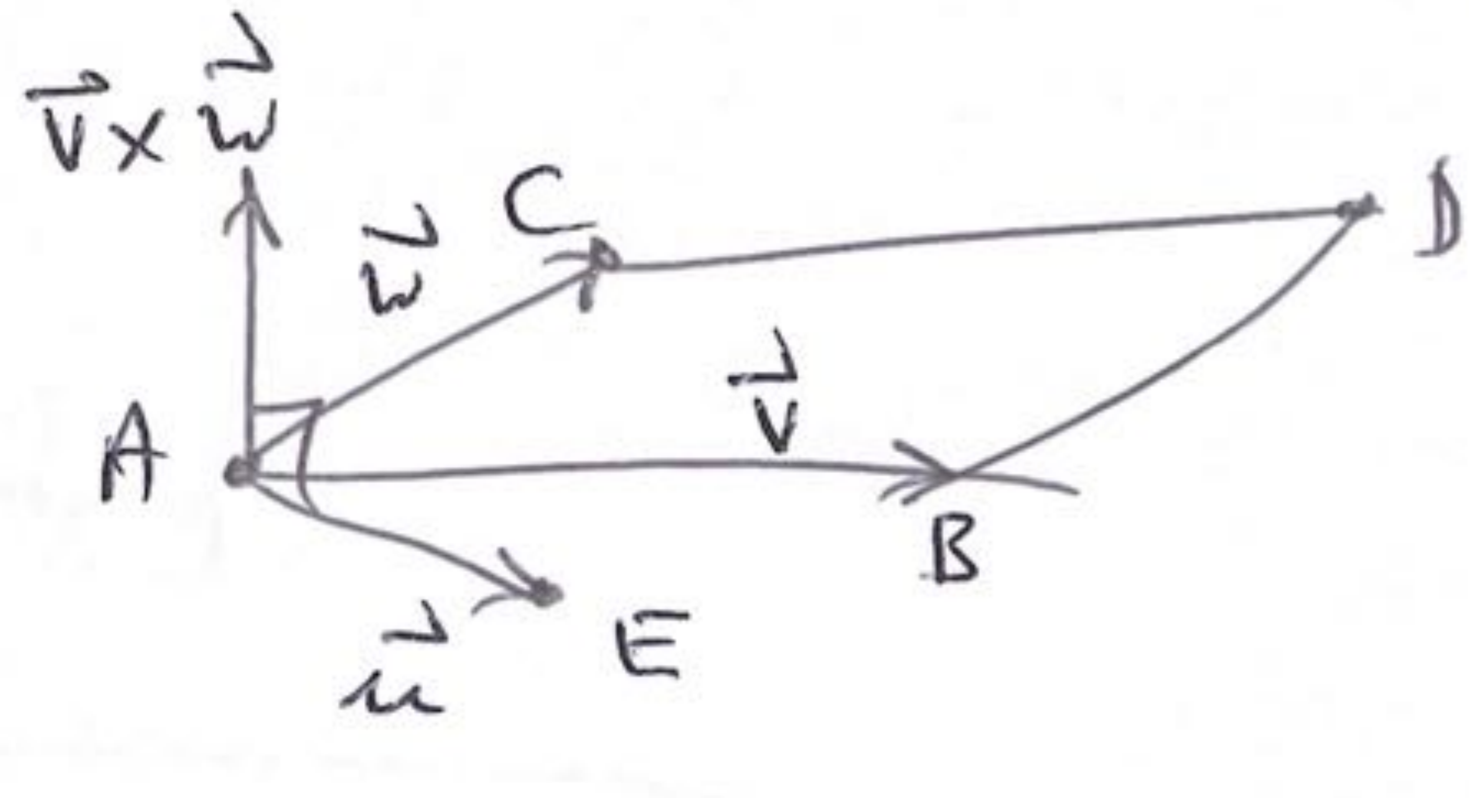


$$D = A + \vec{v} + \vec{w}$$
$$= A + \vec{AB} + \vec{AC}$$
$$= A + B - A + C - A$$
$$= B + C - A$$
$$= (4, 2, -1) + (2, 3, -3) - (3, -1, 1)$$
$$= (3, 6, -5)$$

(b) Find the area of the parallelogram $ABCD$.

$$A = |\vec{v} \times \vec{w}|$$
$$\vec{v} = \vec{AB} = (1, 3, -2)$$
$$\vec{w} = \vec{AC} = (-1, 4, -4)$$
$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -2 \\ -1 & 4 & -4 \end{vmatrix} = (-4, 6, 7)$$
$$A = \sqrt{4^2 + 6^2 + 7^2} = \sqrt{101}$$

(c) Consider the point $E = (4, -5, 5)$. Show that the vector $\overrightarrow{AE}$ lies in the same plane as the parallelogram $ABCD$.



Let $\vec{u} = \vec{AE} = E - A$
$$= (4, -5, 5) - (3, -1, 1)$$
$$\vec{u} = (1, -4, 4)$$

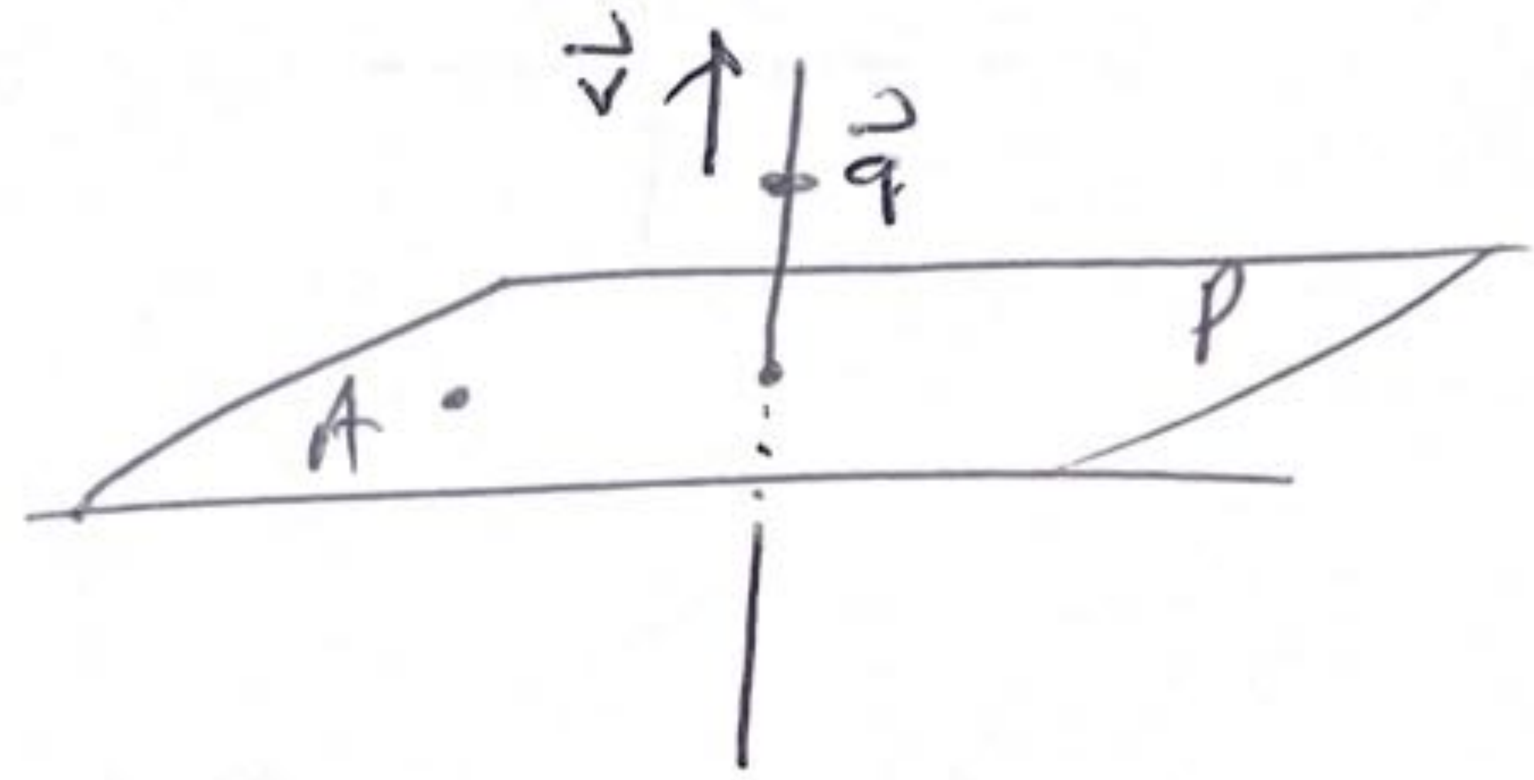
~~Since~~ Just show $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

Well $\vec{u} \cdot (\vec{v} \times \vec{w}) = (1, -4, 4) \cdot (-4, 6, 7) = -4 - 24 + 28 = 0$ ✓

So $\vec{u} \perp (\vec{v} \times \vec{w})$. So $\vec{u}$ lies in plane containing $\vec{v}, \vec{w}$ which is plane containing Π GRAM ABCD

(3) [13 pts] Let $\mathcal{P}$ be the plane through the point $A = (1, 0, 2)$ that is perpendicular to the line with parameterization $\mathbf{r}(t) = \mathbf{q} + t\mathbf{v} = (-1 + 3t)\mathbf{i} + (4 - 2t)\mathbf{j} + 4t\mathbf{k}$.

(a) Draw a schematic diagram showing the relationship between the plane and the line. Include the point, A, and the vectors, $\mathbf{q}$ and $\mathbf{v}$ in your sketch.



(b) Find an equation of the form $Ax + By + Cz = D$ for the plane, $\mathcal{P}$.

$\vec{n} = \vec{v} = (3, -2, 4)$ is a normal to plane. (See PICTURE)

$\vec{x}_0 = (1, 0, 2)$ is a point in plane

Eqn: $0 = (\vec{x} - \vec{x}_0) \cdot \vec{n} = (x-1, y, z-2) \cdot (3, -2, 4)$

$$3(x-1) - 2y + 4(z-2) = 0$$

$$3x - 2y + 4z = 11$$

(c) Find a parameterization of the plane, $\mathcal{P}$.

From (b) $\quad 4z = 11 - 3x + 2y$

$$z = \frac{11}{4} - \frac{3}{4}x + \frac{1}{2}y$$

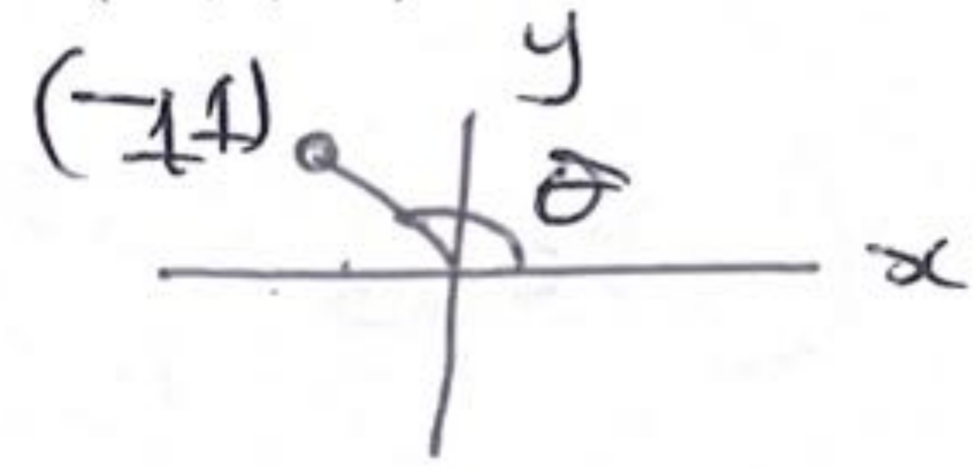
So set
$$\boxed{\begin{aligned} x &= s \\ y &= t \\ z &= \frac{11}{4} - \frac{3}{4}s + \frac{1}{2}t \end{aligned}}$$

or

$$\vec{r}(s,t) = \left(0, 0, \frac{11}{4}\right) + s\left(1, 0, -3/4\right) + t\left(0, 1, \frac{1}{2}\right)$$

(4) [12 pts]
(a) Let P be the point with rectangular coordinates $(x, y, z) = (-1, 1, \sqrt{2})$.

(i) Find the cylindrical coordinates of P.



$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\theta = 3\pi/4$$

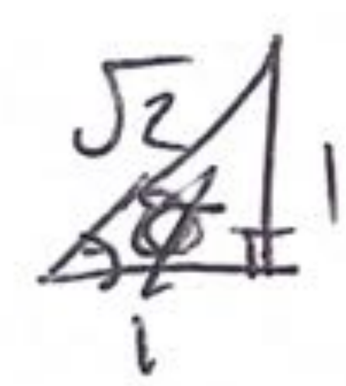
$$z = \sqrt{2}$$

$$(r, \theta, z) = \left(\sqrt{2}, \frac{3\pi}{4}, \sqrt{2}\right)$$

(ii) Find the spherical coordinates of P.

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2 + 1^2 + (\sqrt{2})^2} = 2$$

$\theta = 3\pi/4$ as above

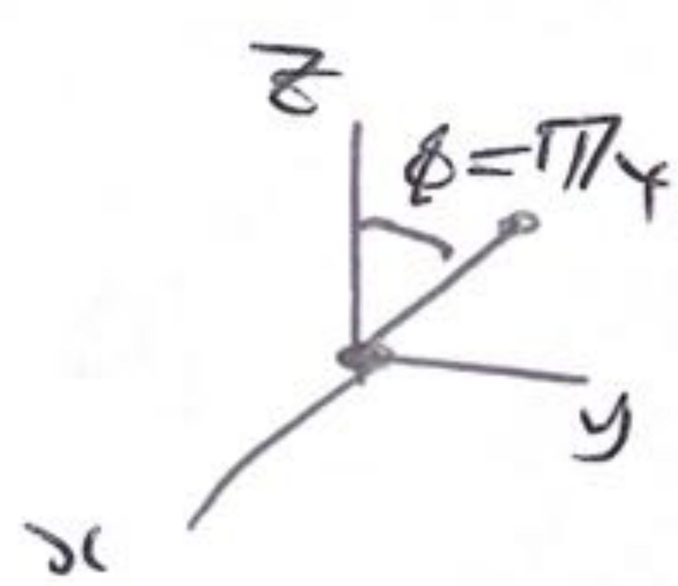


$$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right)$$

$$\phi = \pi/4$$

(b) Convert the equation $\phi = \pi/4$ in spherical coordinates (ρ, θ, ϕ), into an equation involving cylindrical coordinates (r, θ, z).

$$z = \rho\cos\phi = \sqrt{x^2+y^2+z^2}\,\cos\phi$$



$$z = \sqrt{r^2+z^2}\,\cos\phi \qquad \cos\phi = \frac{1}{\sqrt{2}}$$

$$z = \sqrt{r^2+z^2} \cdot \frac{1}{\sqrt{2}}$$

$$2z^2 = r^2 + z^2$$

$$z^2 = r^2$$

$$z = \pm r.$$

But $\phi = \pi/4$ means $z > 0$.
So get

$$\boxed{z = r}$$

(5) [13 pts] Let $(x, y, z) = \mathbf{r}(t) = (t^2 + 3t, e^{2t}, \sin t)$ be the position of a particle at time t.

(a) Find the velocity vector of the particle at time t.

$$\vec{v}(t) = \vec{r}\,'(t) = (2t + 3, 2e^{2t}, \cos t)$$

(b) Find the speed of the particle at time $t = 0$.

$$\vec{v}(0) = (3, 2, 1)$$

$$|\vec{v}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

(c) Find a parametrization for the tangent line to the particle's motion at the point where $t = 0$.

$$\vec{l}(s) = \vec{p} + s\vec{v} = \vec{r}(0) + s\,\vec{r}\,'(0)$$

$$\boxed{\cancel{\vec{l}(s) = (3, 2, 1) + s}}$$

$$\vec{r}(0) = (0, 1, 0)$$

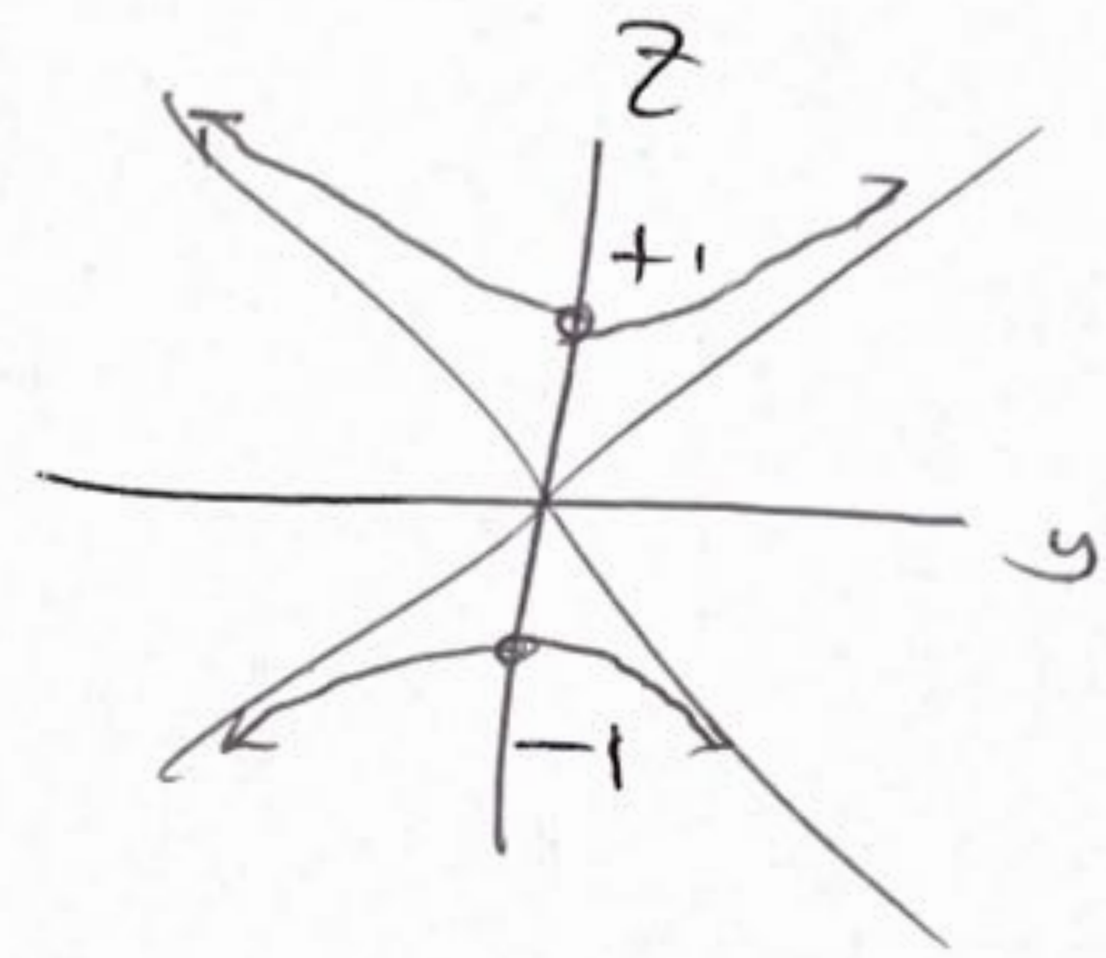
$$\vec{l}(s) = (0, 1, 0) + s(3, 2, 1) = (3s, 1 + 2s, 1)$$

(6) [12 pts] Make labelled sketches of the traces (slices) of the surface $z^2 = 1 + y^2 + 4x^2$ in the planes $x = 0$, $y = 0$, and $z = k$ for $k = 0, \pm 1, \pm 2$. Be sure to include any asymptotes and intercepts in your sketches. Then make a labelled sketch of the surface.

$\boxed{x=0}$ $z^2 - y^2 = 1$

ASYMPTOTES : $z^2 - y^2 = 0 \Rightarrow z = \pm y$

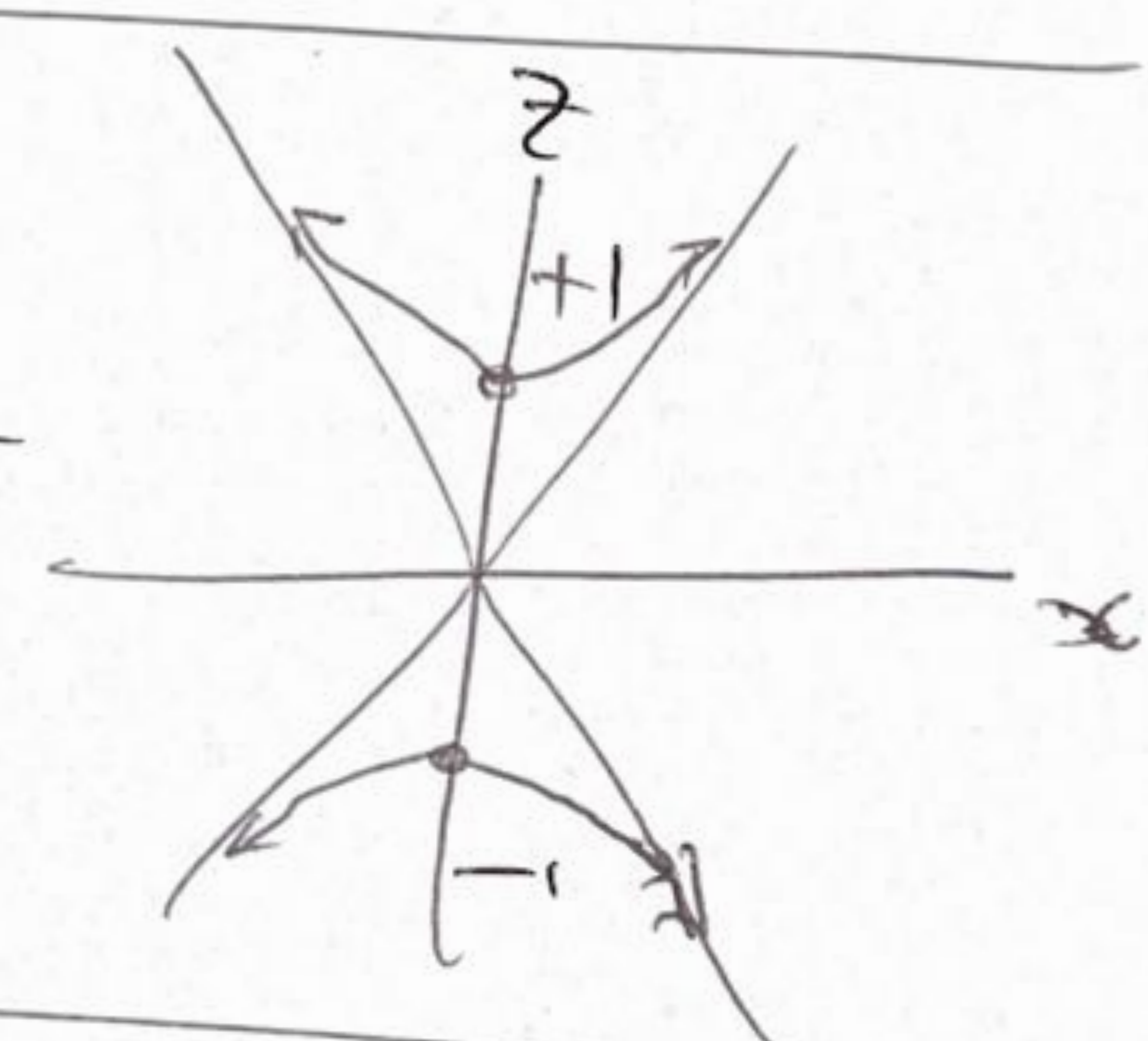
INTERCEPTS $(0, \pm 1)$



$\boxed{y=0}$ $z^2 - 4x^2 = 1$

ASYMPTOTES : $z^2 - 4x^2 = 0 \Rightarrow z = \pm 2x$

INTERCEPTS $(0, \pm 1)$



$\boxed{z=k}$ $4x^2 + y^2 = k^2 - 1$

$k=0$: $4x^2 + y^2 = -1$ NO SOLUTIONS

$k = \pm 1$: $4x^2 + y^2 = 0$ $(0, 0)$

$k = \pm 2$: $4x^2 + y^2 = 3$

ELLIPSE INTERCEPTS

$(0, \pm\sqrt{3})$, $(\pm\sqrt{3}/2, 0)$