LAST NAME:	FIRST NAME:	CIRCLE:	Khoury 5:30pm	Coskunuzer 8:30am
CAYLEY	ARTHUR	Coskunuzer 11:30am	Zweck 1pm	Zweck 4pm

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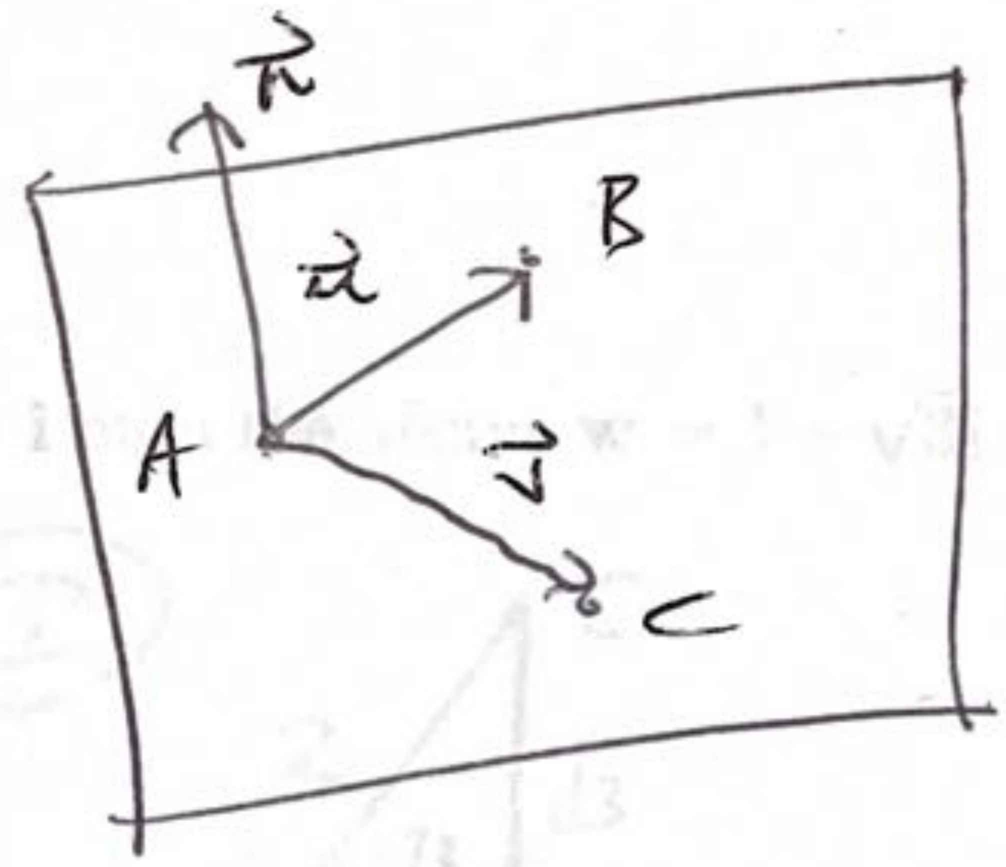
1 /10	2 /10	3 /10	4 /10	5 /10	
6 /10	7 /10	8 /10	9 /10	10 /10	T /100

MATH 2415 Final Exam, Fall 2023

No books or notes! **NO CALCULATORS! Show all work and give complete explanations.** This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts] Find an equation of the form $Ax + By + Cz = D$ for the plane containing the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$. Which of the following points lies on this plane: $P = (2, -1, 3)$, $Q = (3, -4, 3)$.

A B C



$\vec{u} = \vec{AB} = B - A = (1, -1, 0)$

$\vec{v} = \vec{AC} = C - A = (1, 0, -1)$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= (1, 1, 1)$$

$\vec{x}_0 = A = (0, 1, 1)$

So $(\vec{x} - \vec{x}_0) \cdot \vec{n} = 0$ gives

$(x, y-1, z-1) \cdot (1, 1, 1) = 0$

$\boxed{x + y + z = 2}$

$\underline{\underline{P:}}$

$2 - 1 + 3 = 4 \neq 2$

P DOES NOT LIE ON PLANE

$\underline{Q}$ $3 - 4 + 3 = 2$

Q DOES LIE ON PLANE

(2) [10 pts]
(a) Find the volume of the parallelipiped determined the the vectors $\mathbf{u} = (1, 4, 2)$, $\mathbf{v} = (-1, 1, 4)$ and $\mathbf{w} = (5, 1, 2)$.

$$\text{VOL} = |(\vec{u} \times \vec{v}) \circ \vec{w}| = \left| \begin{vmatrix} 1 & 4 & 2 \\ -1 & 1 & 4 \\ 5 & 1 & 2 \end{vmatrix} \right|$$

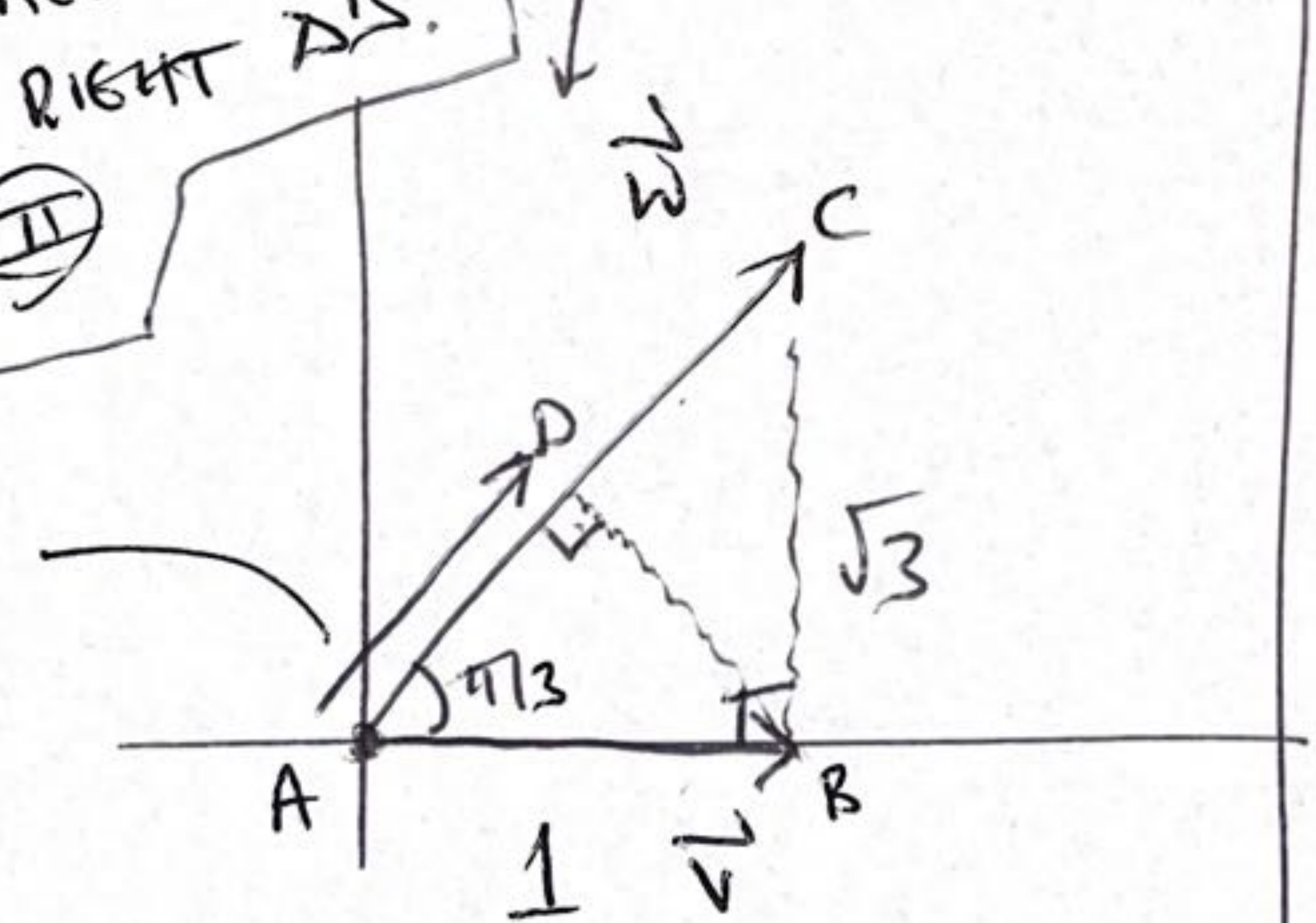
$$= |-2 - 4(-22) + 2(-6)| \qquad = \cancel{|-102|}$$

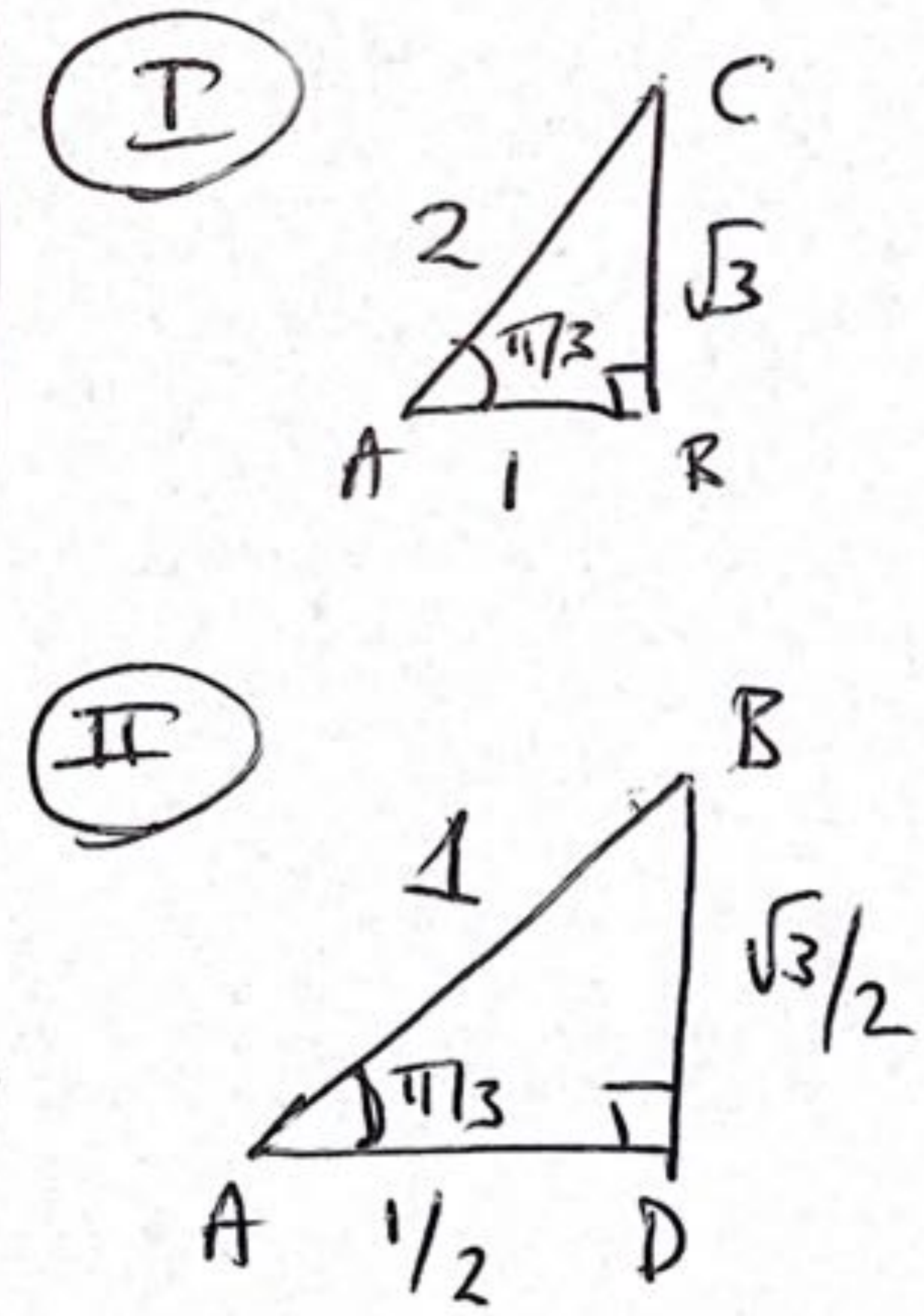
$$= 74 \qquad = \cancel{102}$$

(b) Make a sketch that shows how to project the vector $\mathbf{v} = \mathbf{i}$ onto the vector $\mathbf{w} = \mathbf{i} + \sqrt{3}\mathbf{j}$. Use your sketch to find the component of $\mathbf{v}$ in the direction $\mathbf{w}$.

THIS DIAGRAM HAS 2 RIGHT Δ'S
(I) + (II)

$\text{PROJ}_{\vec{w}}(\vec{v})$





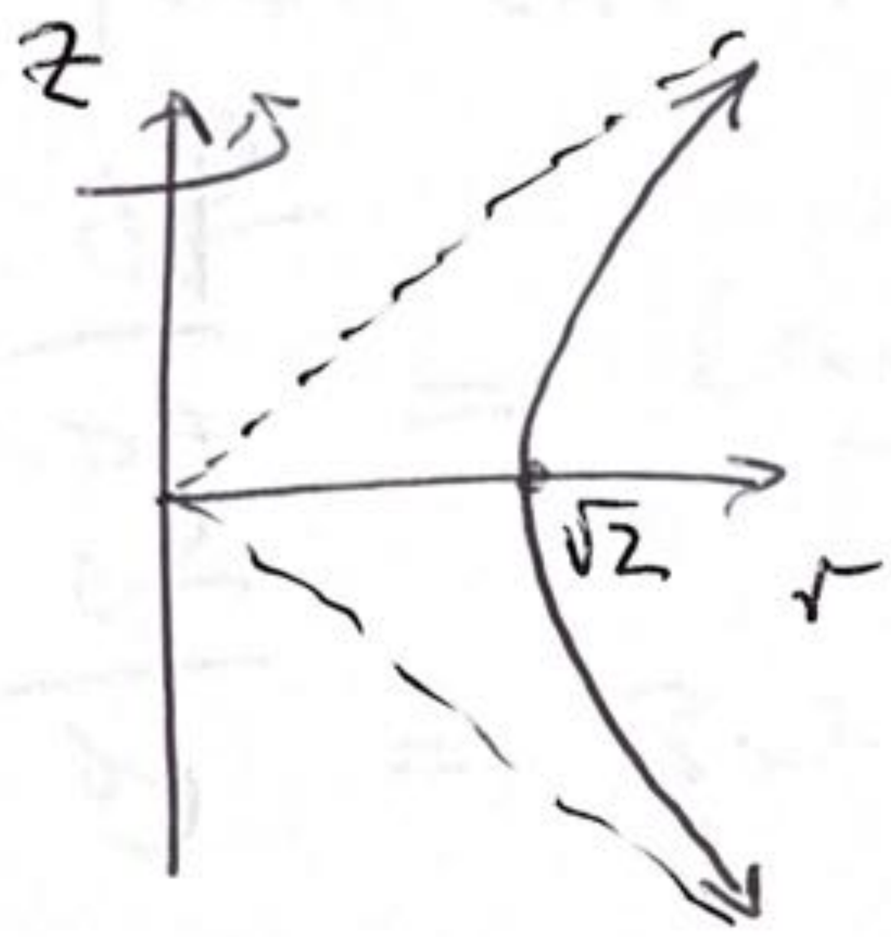
$\text{COMP}_{\vec{w}}(\vec{v}) = +|\vec{AD}|$ AS $\text{PROJ}_{\vec{w}}(\vec{v})$ POINTS IN SAME DIR^N AS $\vec{w}$

$= \frac{1}{2}$ FROM TRIANGLE (II)

(3) [10 pts]
(a) Sketch the surface $x^2 + y^2 - z^2 = 2$ for $\cancel{0}$ $-\sqrt{2} \le z \le \sqrt{2}$.

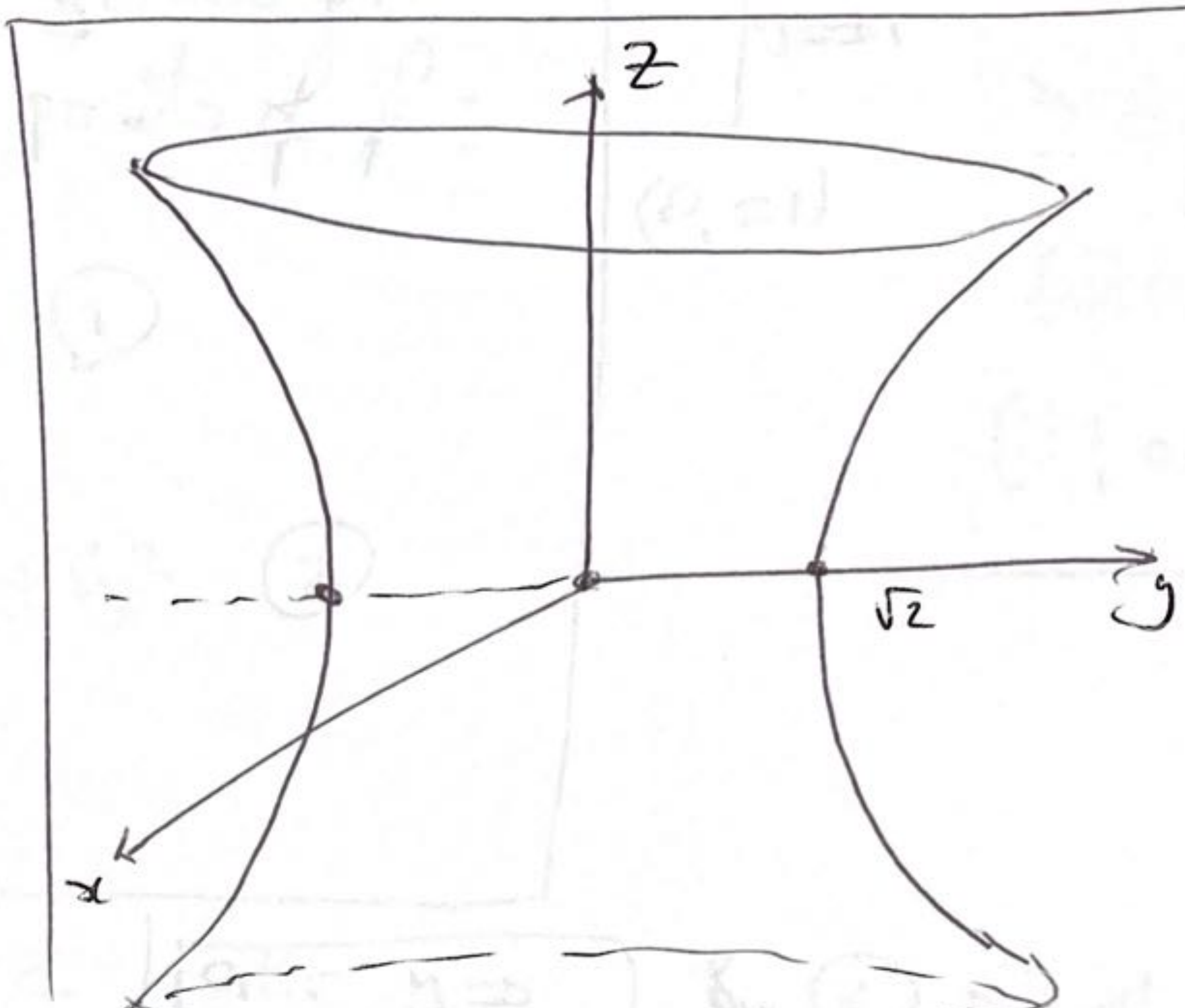
THIS IS A SURFACE OF REVOLUTION, IN CYL COORDS:

$r^2 - z^2 = 2$



$z=0$: $r = \sqrt{2}$

ASYMPTOTES: $z = \pm r$



(b) Show that the line through the point $(1, 1, 0)$ in the direction of the vector $(-1, 1, \sqrt{2})$ lies on the surface in (a).

Parametrize the line:

$$\vec{r}(t) = \vec{p} + t\vec{v} = (1,1,0) + t(-1,1,\sqrt{2})$$

$$(x,y,z) = \vec{r}(t) = (1-t,\ 1+t,\ \sqrt{2}t)$$

To show line lies on surface plug

$x = 1-t$, $y = 1+t$, $z = \sqrt{2}t$ into $x^2+y^2-z^2=2$.

Well $x^2+y^2-z^2$

$$= (1-t)^2 + (1+t)^2 - (\sqrt{2}t)^2$$

$$= 1 - 2t + t^2 + 1 + 2t + t^2 - 2t^2$$

$$= 2 \checkmark$$

(4) [10 pts] Find and classify all critical points of the function

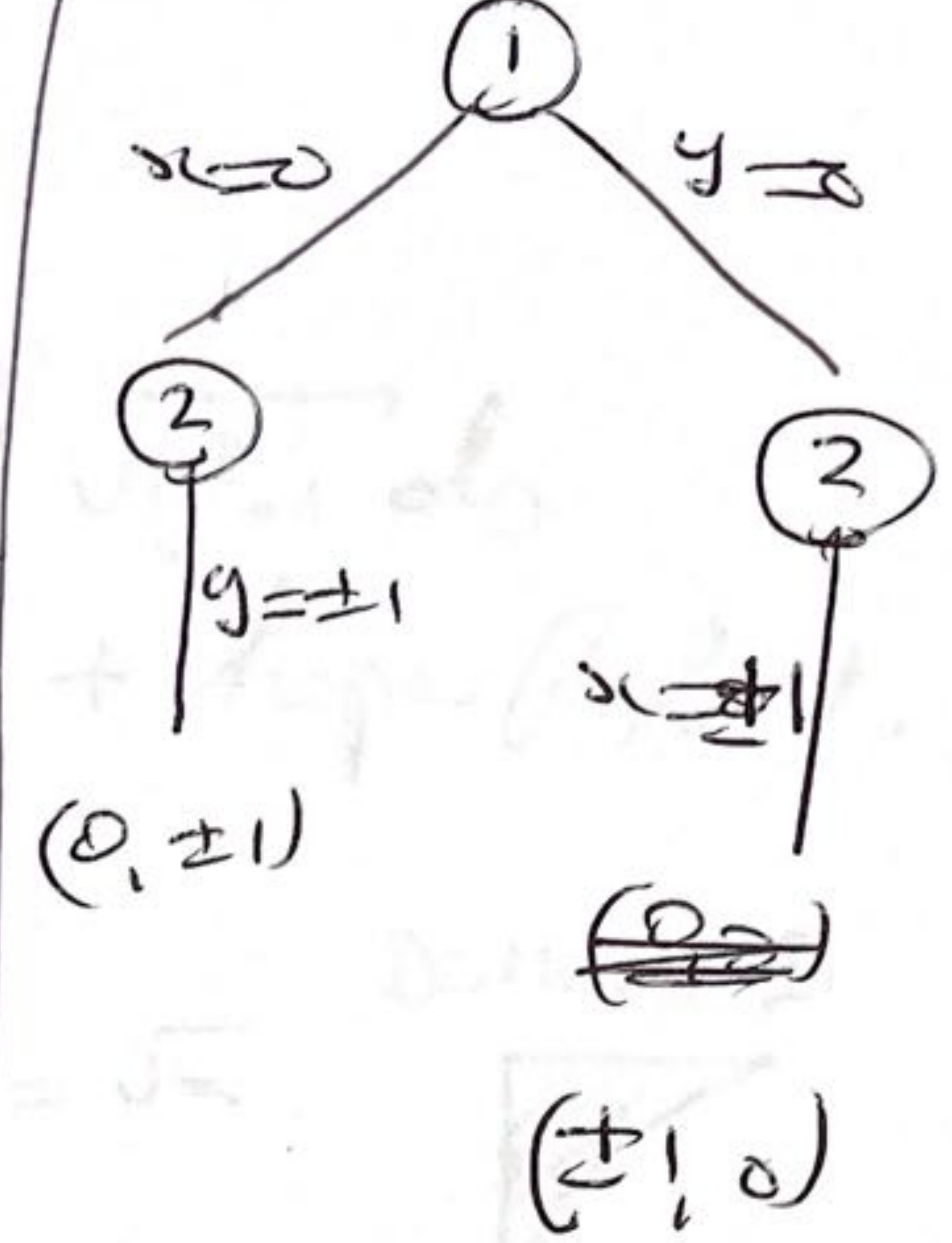
$$f(x,y) = 3x^2y - 3y + y^3$$

This is a local optimization problem.
First find critical points of f:

$$0 = \frac{\partial f}{\partial x} = 6xy \quad (1)$$

$$0 = \frac{\partial f}{\partial y} = 3x^2 - 3 + 3y^2 \quad (2)$$

By (1) $x = 0$ or $y = 0$



$\boxed{\text{CASE } x=0}$ By (2) $y^2 = 1$

$y = \pm 1$

$(x,y) = (0, \pm 1)$

$\boxed{\text{CASE } y=0}$ By (2) $x^2 = 1$

So $x = \pm 1$

$(x,y) = (\pm 1, 0)$

$\boxed{\text{2ND DER TEST}}$

$$D = \text{DET}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \text{DET}\begin{bmatrix} 6y & 6x \\ 6x & 6y \end{bmatrix} = 36(y^2 - x^2)$$

(x,y)	D	f_{xx}	CLASSIFICATION
~~(0,0)~~	~~0~~		
$(0,1)$	$36>0$	$6>0$	LOCAL MIN
$(0,-1)$	$36>0$	$-6<0$	LOCAL MAX
$(1,0)$	$-36<0$		SADDLE PT
$(-1,0)$	$-36<0$		SADDLE PT

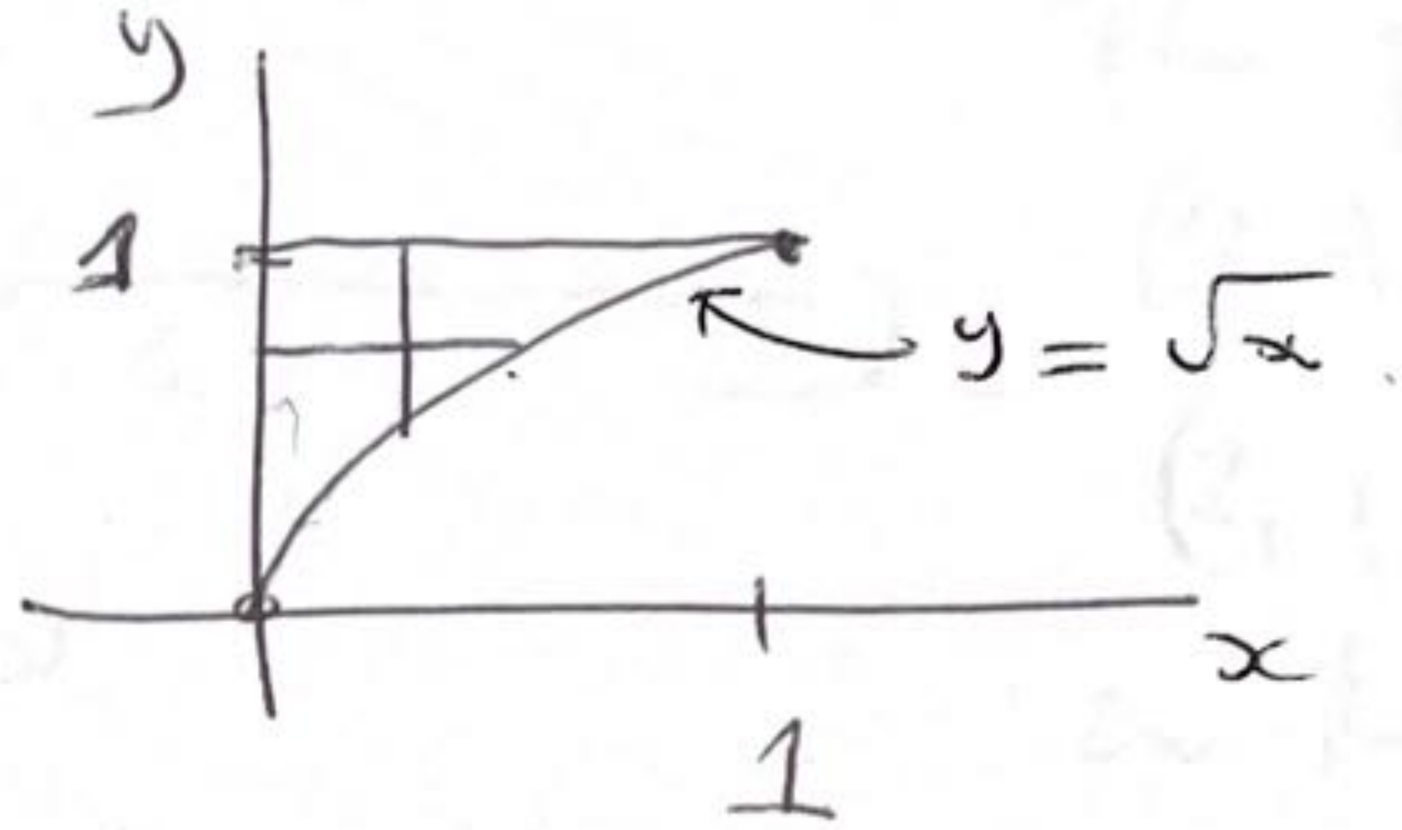
(5) [10 pts] Evaluate the double integral $\int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \sqrt{y^3+1}\, dy dx.$ $= I$

Since I have no idea how to do $\int \sqrt{y^3+1}\, dy$ lets switch order of integration + hope for best.

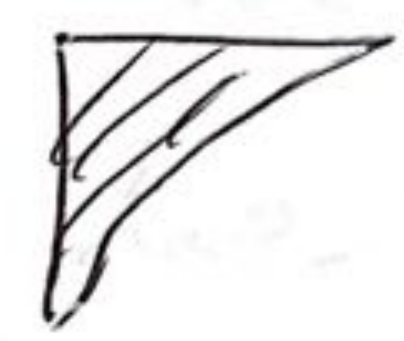
$0 \le x \le 1$

$\sqrt{x} \le y \le 1$ IS

TYPE I



DOMAIN D



IS BOTH TYPE I + II

AS TYPE II REGION:

$0 \le y \le 1$

$0 \le x \le y^2$

So

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=y^2} \sqrt{y^3+1}\, dx\, dy$$

$$= \int_{y=0}^{y=1} \left[\sqrt{y^3+1}\; x \right]_{x=0}^{x=y^2}$$

$$= \int_{y=0}^{y=1} \sqrt{y^3+1}\; y^2\, dy$$

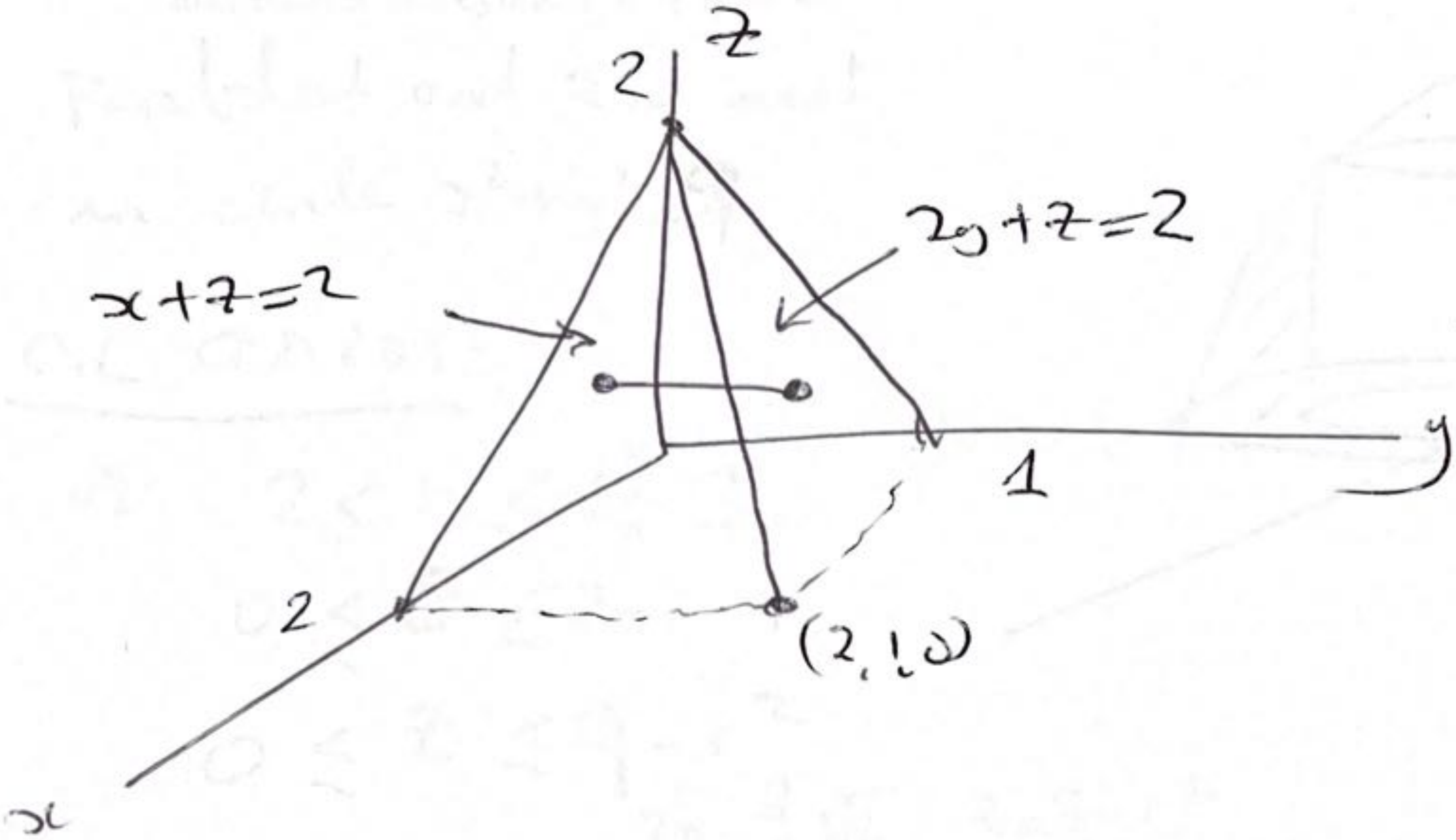
SUB $u = y^3+1$

$du = 3y^2 dy$

$$= \frac{1}{3} \int_{u=1}^{u=2} \sqrt{u}\, du = \frac{1}{3}\left[\frac{2}{3} u^{3/2}\right]_{u=1}^{u=2}$$

$$= \boxed{\frac{2}{9}\left(2^{3/2} - 1\right)}$$

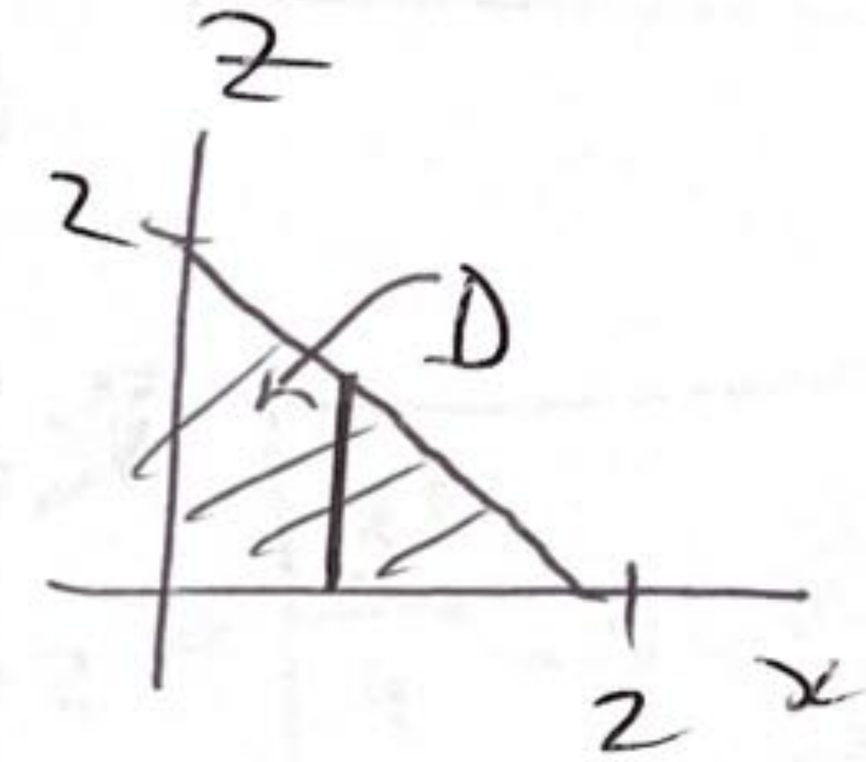
(6) [10 pts] Let E be the solid in the first octant that is bounded by the planes $x + z = 2$ and $2y + z = 2$. Calculate $\iiint_E z\,dV$. *Hint:* It may be helpful to sketch E.



THESE 2 PLANES meet in a line. The points $(0,0,2)$ and $(2,1,0)$ lie on the line

USE STICKS GO LEFT—RIGHT:

SHADOW OF STICKS ON xz—PLANE IS



$$\left.\begin{array}{l} 0 \le x \le 2 \\ 0 \le z \le 2-x \end{array}\right\} D$$

$0 \le y \le 1 - \frac{z}{2}$ as $2y+z=2$ is RIGHT SURFACE

So

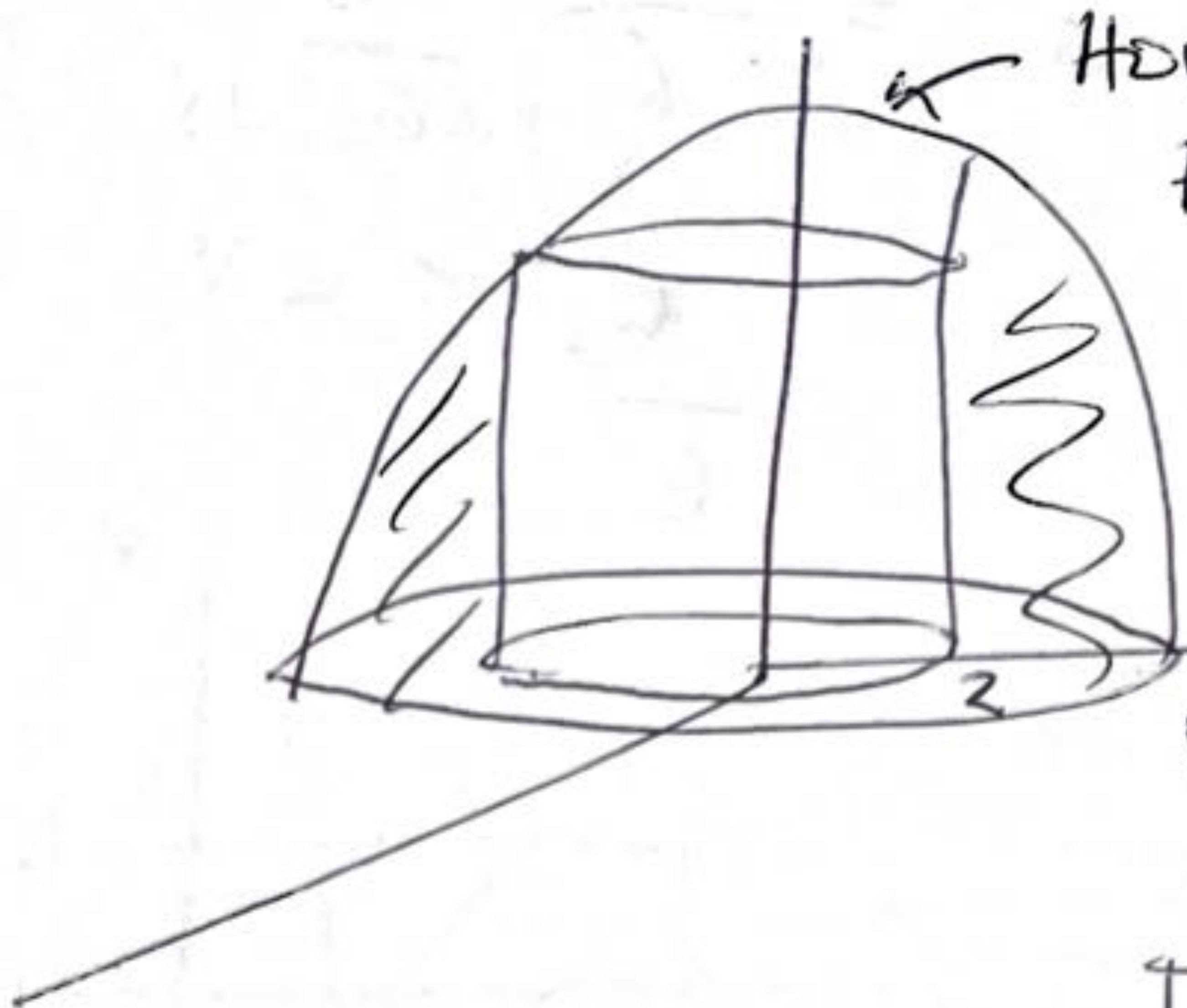
$$\iiint_E z\,dV = \int_{x=0}^{x=2} \int_{z=0}^{z=2-x} \int_{y=0}^{y=1-z/2} z \, dy\, dz\, dx$$

$$= \int_{x=0}^{x=2} \int_{z=0}^{z=2-x} z\left(1 - z/2\right) dz\, dx = \int_{x=0}^{x=2} \left[\frac{z^2}{2} - \frac{z^3}{6}\right]_{z=0}^{z=2-x}$$

$$= \int_0^2 \left(\frac{(2-x)^2}{2} - \frac{(2-x)^3}{6}\right) dx = \left[-\frac{(2-x)^3}{6} + \frac{(2-x)^4}{24}\right]_0^2 = \boxed{2/3}$$

(7) [10 pts]
(a) Find $\iiint_E \sqrt{x^2+y^2}\,dV$ where E is the solid above the xy-plane, below the paraboloid $z = 9 - x^2 - y^2$ and *outside* the cylinder $x^2+y^2=4$.

Paraboloid and $z=0$ meet in circle $x^2+y^2=9$



HORRIBLE PICTURE BUT I AM ON A BUMPY AIRPLANE!

CYL COORDS:

$2 \le r \le 3$

$0 \le \theta \le 2\pi$

$0 \le z \le 9 - r^2$

$\cancel{2\pi(128}$

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_{\theta=0}^{2\pi}\int_{r=2}^{3}\int_{z=0}^{z=9-r^2} r \cdot r\,dz\,dr\,d\theta$$

$$= 2\pi\int_{r=2}^{3} r^2(9-r^2)\,dr = 2\pi\left[\frac{9}{3}r^3 - \frac{1}{5}r^5\right]_2^3 = \cancel{\frac{3^5}{4} - 36 + \frac{32}{3}}$$

$$= 2\pi\left[81 - \frac{3^5}{5} - 24 + \frac{32}{5}\right]$$

$$= \frac{148\pi}{5}$$

(b) Use a triple integral in spherical coordinates to find the volume of the sphere of radius R.

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$0 \le \rho \le R$

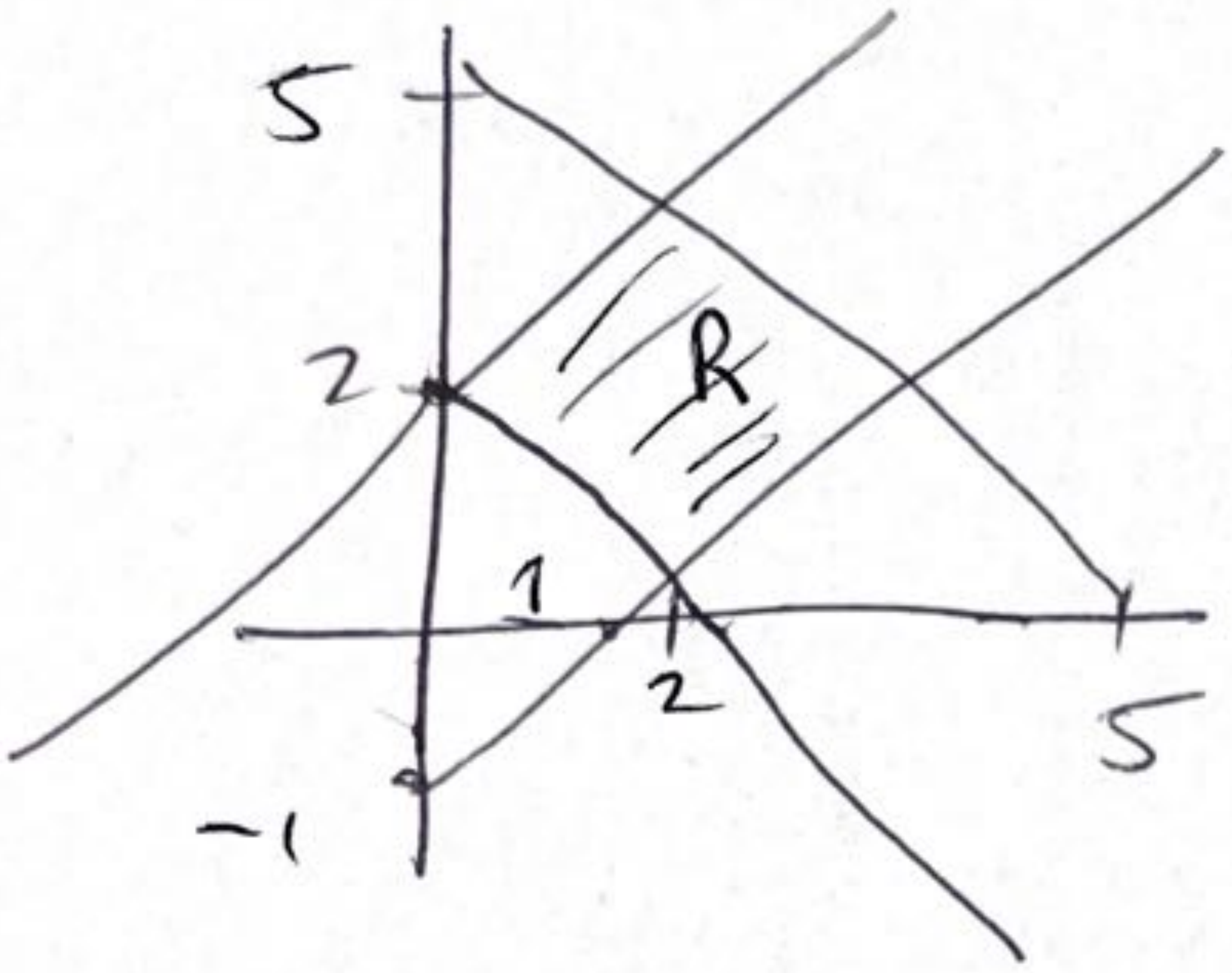
$dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

√√

$$\text{VOL} = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{R} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \boxed{\frac{4\pi R^3}{3}}$$

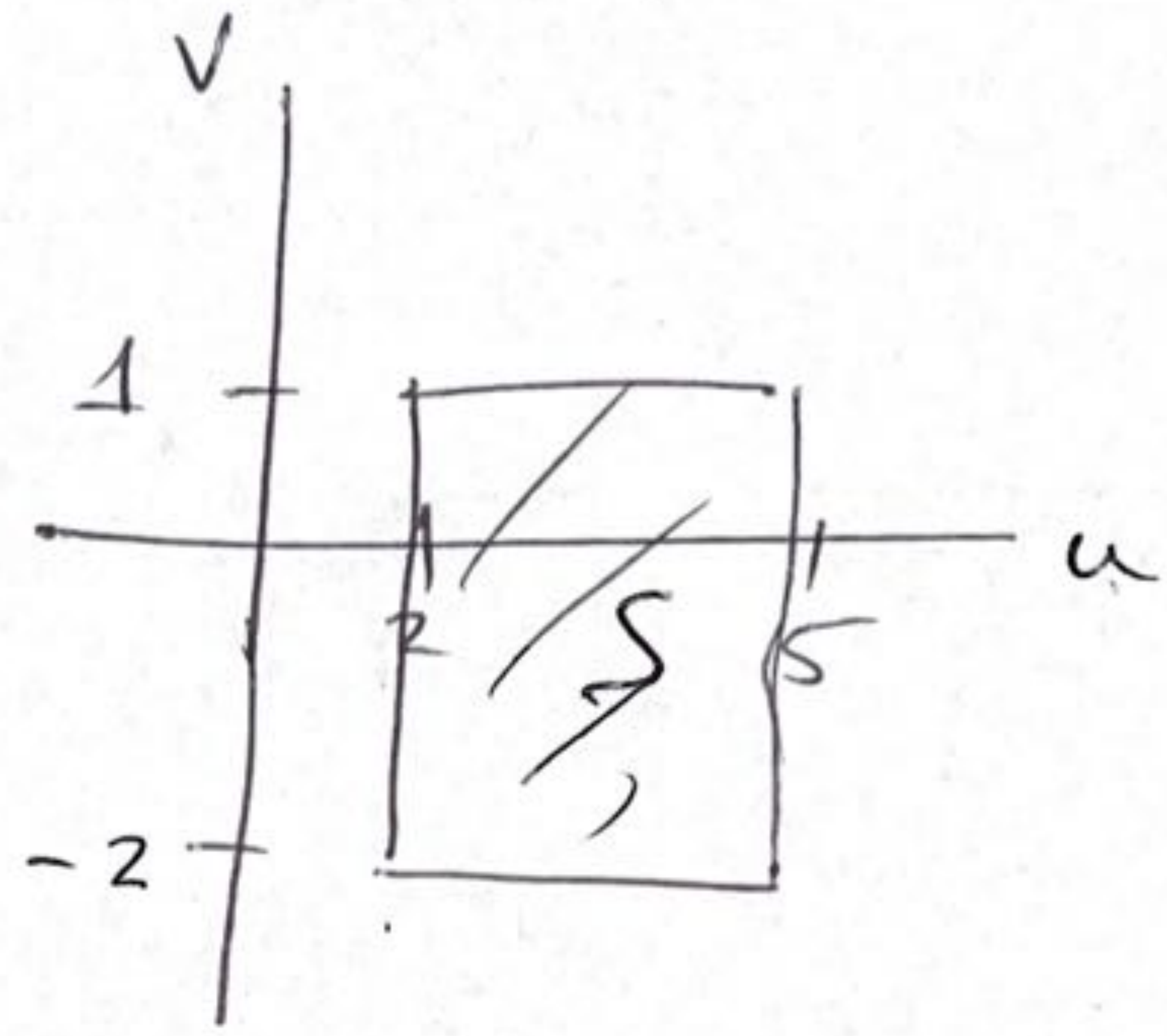
$$= 2\pi\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^R \rho^2\,d\rho\right) = 2\pi\cdot(2)\left(\frac{\rho^3}{3}\right)\Bigg]_{\rho=0}^{\rho=R}$$

(8) [10 pts] Use the change of variables theorem to evaluate $\iint_R (x+y)^2 e^{x-y}\, dxdy$ where R is the parallelogram bounded by $x+y=2$, $x+y=5$, $x-y=-2$ and $x-y=1$.



Let $u = x+y$

$v = x-y$



S is

$2 \le u \le 5$

$-2 \le v \le 1$

$u+v = 2x$

So $x = \frac{u+v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \text{DET}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{2}$$

$u - v = 2y$

So $y = \frac{u-v}{2}$

$$\boxed{\text{So } I = \int_{u=2}^{5}\int_{v=-2}^{1} u^2 e^v \left|-\tfrac{1}{2}\right| dv\,du}$$

$$= \frac{1}{2}\left(\int_2^5 u^2 du\right)\left(\int_{-2}^{1} e^v dv\right)$$

$$= \frac{1}{2}\left(\frac{5^3}{3} - \frac{2^3}{3}\right)\left(e^1 - e^{-2}\right)$$

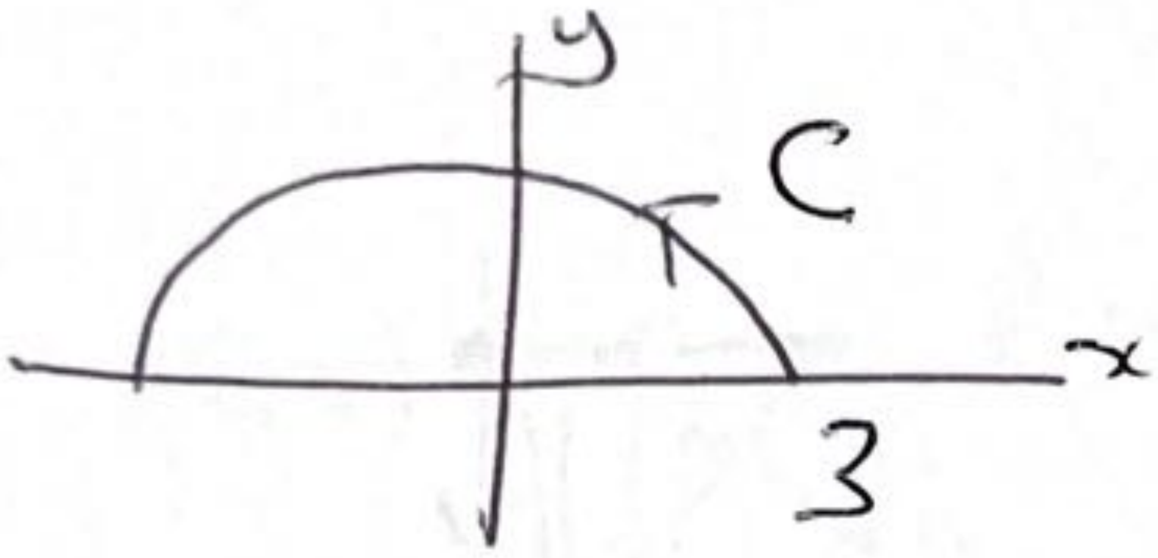
$$= \frac{117}{6}\left(e - e^{-2}\right)$$

(9) [10 pts]

(a) Let C be the semicircle $x^2 + y^2 = 9$ with $y \geq 0$ oriented counter-clockwise and let $\mathbf{F}$ be the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\vec{r}(t) = (3\cos t, 3\sin t), \quad 0 \leq t \leq \pi$$

$$\vec{r}\,'(t) = (-3\sin t, 3\cos t)$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\pi} (-3\sin t, 3\cos t) \cdot (-3\sin t, 3\cos t)\,dt$$

$$= \int_0^{\pi} 9\sin^2 t + 9\cos^2 t\,dt = \boxed{9\pi}$$

(b) Let C be the curve $y = \sqrt[3]{x+1}$ from $(-1, 0)$ to $(0, 1)$. Evaluate

$$\int_C \underset{P}{(2xy + 3)}dx + \underset{Q}{(x^2 + 10y)}dy$$

$$\frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial P}{\partial y} = 2x$$

So $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ so $\vec{F} = \nabla f$ IS CONSERVATIVE.

$\boxed{\text{FIND } f}$

$$\frac{\partial f}{\partial x} = P \Rightarrow f(x,y) = \int (2xy + 3)\,dx = x^2 y + 3x + g(y)$$

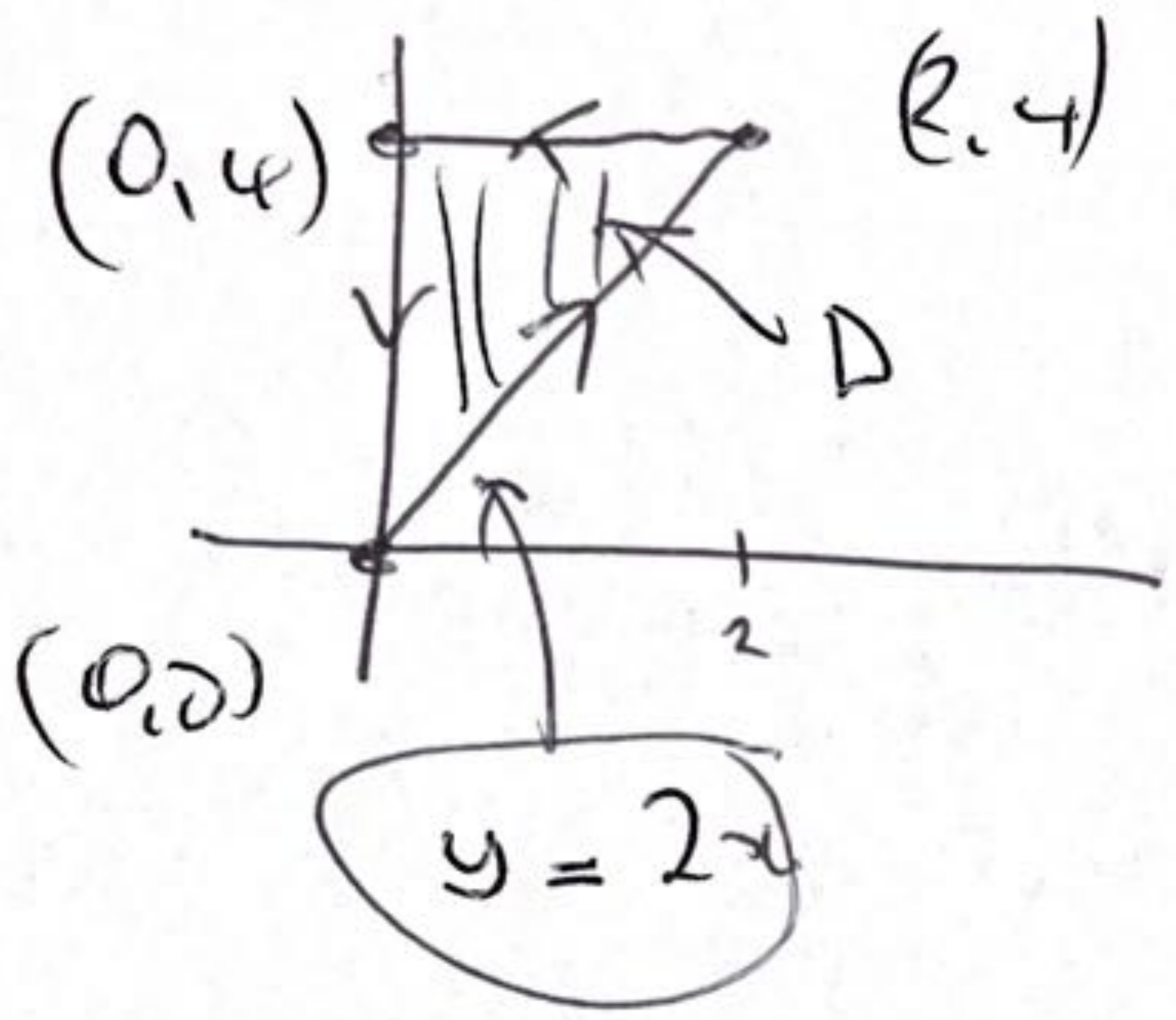
$$\frac{\partial f}{\partial y} = Q \Rightarrow f(x,y) = \int (x^2 + 10y)\,dy = x^2 y + 5y^2 + h(x)$$

So $f(x,y) = x^2 y + 3x + 5y^2 + C$ WORKS.

FTC: $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(0,1) - f(-1,0) = 5 - (-3) = \boxed{8}$

(10) [10 pts] Let D be the triangular domain with vertices $(0,0), (0,4)$ and $(2,4)$. Let C be the boundary of D, oriented counter clockwise. Evaluate

$$I = \int_C \overset{P}{(\sqrt{x^3-1}+2xy^2)}\,dx + \overset{Q}{(x^2y-e^y)}\,dy.$$



$C = \partial D.$

D is Type I

$0 \le x \le 2$

$2x \le y \le 4$

By Green's Thm

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

So

$$I = \iint_D (2xy - 4xy)\,dA = -2\iint_D xy\,dA$$

$$= -2\int_{x=0}^{x=2}\int_{y=2x}^{y=4} xy\,dy\,dx = -2\int_{x=0}^{x=2} x\left[\frac{y^2}{2}\right]_{y=2x}^{y=4} dx$$

$$= -2\int_{x=0}^{x=2} x(8-2x^2)\,dx$$

$u = 8-2x^2$

$du = -4x\,dx$

$$= \frac{1}{2}\int_{u=8}^{u=0} u\,du = \left[\frac{u^2}{4}\right]_8^0 = \boxed{-16}$$