LAST NAME:	FIRST NAME:	CIRCLE:	Khoury 5:30pm	Coskunuzer 8:30am
LAGRANGE	JOSEPH-LOUIS	Coskunuzer 11:30am	Zweck 1pm	Zweck 4pm

1736 – 1813

MATH 2415 [Fall 2023] Exam II

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. This 75 minute exam is worth 75 points. **Points will be recorded on the top of the second page.**

(1) [10 pts] Suppose that $z = f(x, y) = \cos(3x + 2y)$ where $x = x(t)$ and $y = y(t)$. If $x(0) = \pi/3$, $y(0) = -\pi/4$, $x'(0) = 4$, and $y'(0) = 5$, find $\frac{dz}{dt}$ at $t = 0$.

$z = f(\vec{r}(t))$ with $\vec{r}(t) = (x(t), y(t))$

$\frac{dz}{dt}(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$ By CHAIN RULE FOR FUNCTIONS ON CURVES.

$\vec{r}(0) = (x(0), y(0)) = (\pi/3, -\pi/4)$

$\vec{r}\,'(0) = (x'(0), y'(0)) = (4, 5)$

$\nabla f(x, y) = (-3\sin(3x+2y), -2\sin(3x+2y))$

$\nabla f(\vec{r}(0)) = (-3, -2) \sin\left(3\frac{\pi}{3} + 2(-\pi/4)\right)$

$= (-3, -2) \sin(\pi - \pi/2)$

$= (-3, -2)$ as $\sin \pi/2 = 1$

So $\frac{dz}{dt}(0) = (-3, -2) \cdot (4, 5) = -12 - 10 = \underline{\underline{-22}}$

1 /10	2 /14	3 /13	4 /12	5 /13	6 /13	T /75

(2) [14 pts]
Let $f(x, y) = x^2y^2 - 2x - 2y$ and let $\mathbf{x}_0 = (2, 1)$.

(a) Find the gradient of f at $\mathbf{x}_0$.

$$\nabla f = (2xy^2 - 2,\ 2x^2y - 2)$$

$$\nabla f(2,1) = (2, 6)$$

(b) Find the directional derivative of f at $\mathbf{x}_0$ in the direction of the vector $(4, 3)$.

$$\vec{u} = \frac{(4,3)}{\|(4,3)\|} = \left(\frac{4}{5}, \frac{3}{5}\right)$$

$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u} = (2,6) \cdot \left(\frac{4}{5}, \frac{3}{5}\right) = \frac{26}{5}$$

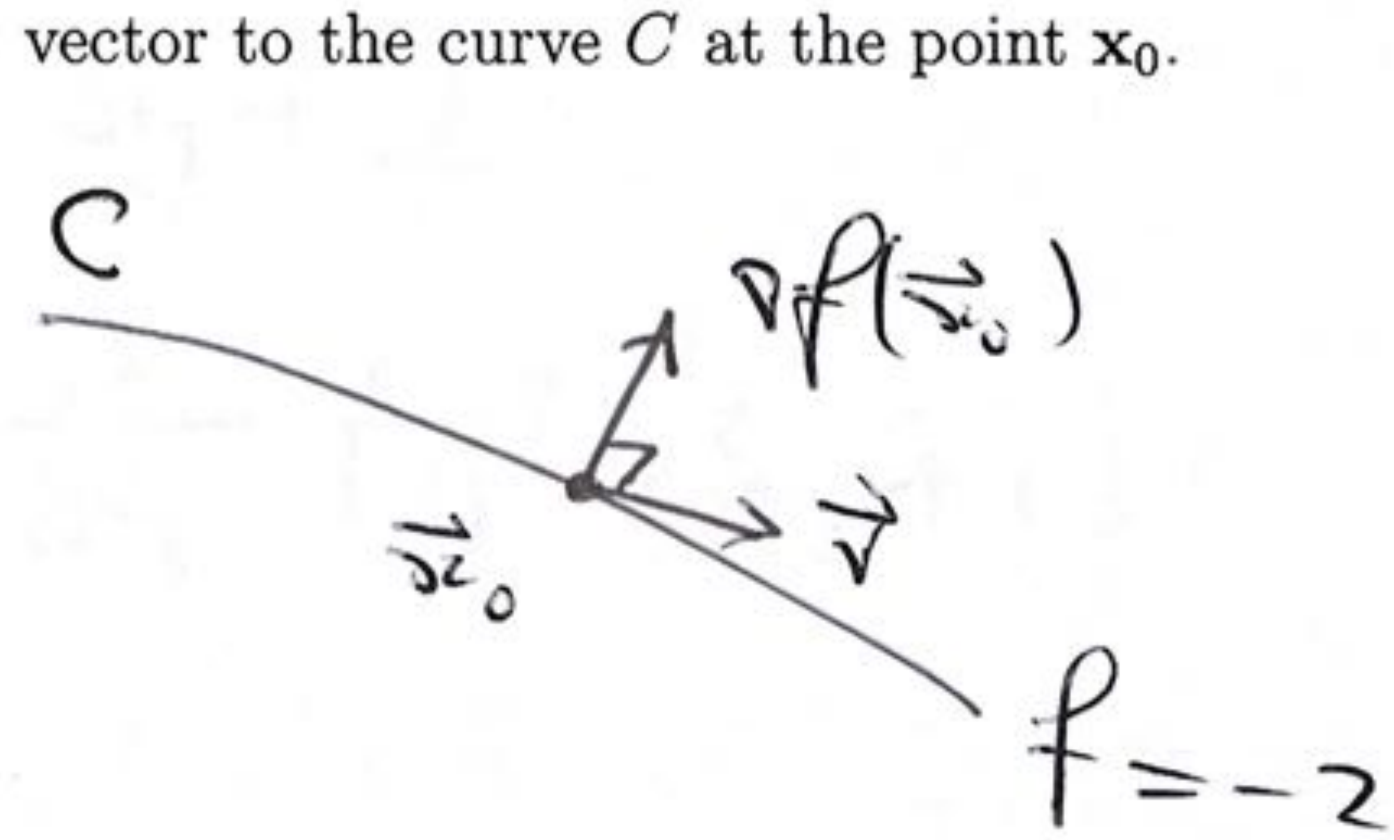
(c) Find the maximum rate of change of f at $\mathbf{x}_0$ and the direction in which it occurs.

MAX RATE OF CHANGE $= |\nabla f(\vec{x}_0)| = |(2,6)| = \sqrt{2^2+6^2} = \sqrt{40}$

DIR^n $= \dfrac{\nabla f(\vec{x}_0)}{|\nabla f(\vec{x}_0)|} = \dfrac{1}{\sqrt{40}}(2,6)$

(d) Let C be the level curve $f(x, y) = -2$. Find a tangent vector to the curve C at the point $\mathbf{x}_0$.

$f(2,1) = 4 \times 1 - 4 - 2 = -2.$



$\vec{v} \perp \nabla f(2,1)$

$\vec{v} \perp (2,6)$

Choose $\boxed{\vec{v} = (-6, 2)}$ OR $(2, -6)$ or $(1, -3)$ ETC

(3) [13 pts] (a) Let $z = f(x,y) = x^2 + 4y^2$. By calculating an equation for the tangent plane to the graph of f at an appropriate point, find an approximation to $f(1.99, 0.99)$.

$(x_0,y_0) = \vec{x}_0 = (2,1)$.

$f(2,1) = 4+4 = 8$ $\quad \frac{\partial f}{\partial y} = 8y = 8$ @ $(2,1)$

$\frac{\partial f}{\partial x} = 2x = 4$ @ $(2,1)$

TANGENT PLANE EQN

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$\boxed{z = 8 + 4(x-2) + 8(y-1)}$$

So $f(1.99, 0.99) \simeq 8 + 4(1.99-2) + 8(0.99-1)$

$= 8 - 0.01 \times 4 - 0.01 \times 8 = 8 - 0.12$

$= 7.88$

(b) Suppose that $z = f(x,y)$ is a function such that $\frac{\partial f}{\partial x}(4,y) = y^2 + 3y + 1$. Let $g(x) = \frac{\partial f}{\partial y}(x,5)$. What is the rate of change of g at $x = 4$? [Do not attempt to find a formula for f].

$$g'(4) = \frac{\partial^2 f}{\partial x \partial y}(4,5) = \frac{\partial^2 f}{\partial y \partial x}(4,5)$$

as mixed partial derivatives commute

NOW $\quad \frac{\partial f}{\partial x}(4,y) = y^2 + 3y + 1$

So $\quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(4,y) = \frac{\partial}{\partial y}(y^2+3y+1)$

$= 2y + 3$

Plug in $y = 5$ to get $\quad g'(4) = \underline{\underline{13}}$

(4) [12 pts] Consider the parameterized surface

$$(x, y, z) = \mathbf{r}(\theta, \phi) = (3\cos\theta\,\sin\phi, 3\sin\theta\,\sin\phi, 3\cos\phi) \qquad \text{for } 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/2.$$

(a) Find an equation of the form $F(x, y, z) = 0$ for this surface.

METHOD I

$x = 3\cos\theta\sin\phi$

$y = 3\sin\theta\sin\phi$

$z = 3\cos\phi$

is just spherical coords with $\rho = 3$.

Now $\rho = \sqrt{x^2+y^2+z^2}$

So our equation is

$\boxed{x^2+y^2+z^2 = 9.}$

METHOD II

$x^2+y^2 = 3^2\cos^2\theta\sin^2\phi + 3^2\sin^2\theta\sin^2\phi$

$= 9(\cos^2\theta + \sin^2\theta)\sin^2\phi$

$= 9\sin^2\phi$

So

$x^2+y^2+z^2 = 9\sin^2\phi + 9\cos^2\phi$

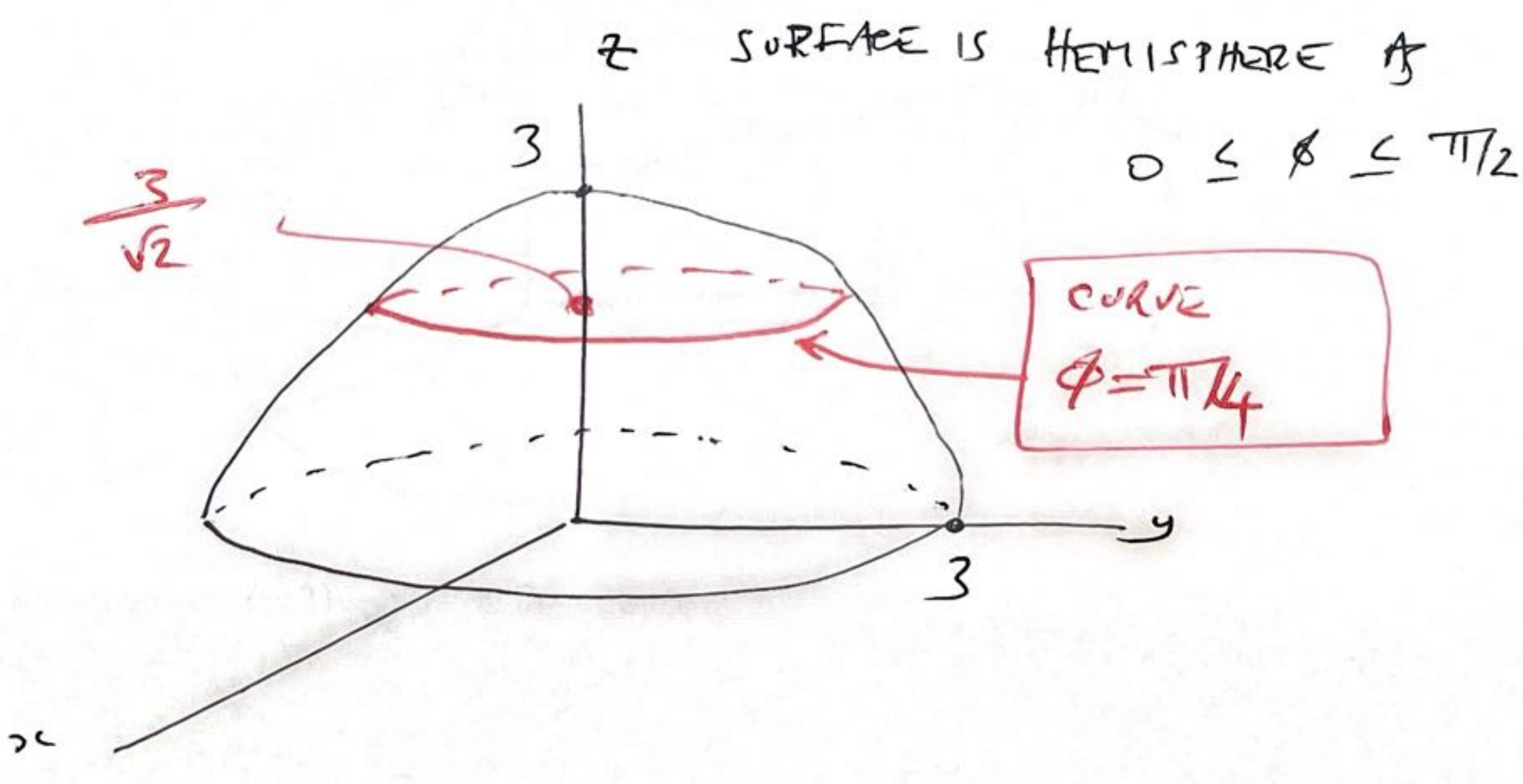
$= 9$

Get $\boxed{x^2+y^2+z^2 = 9}$

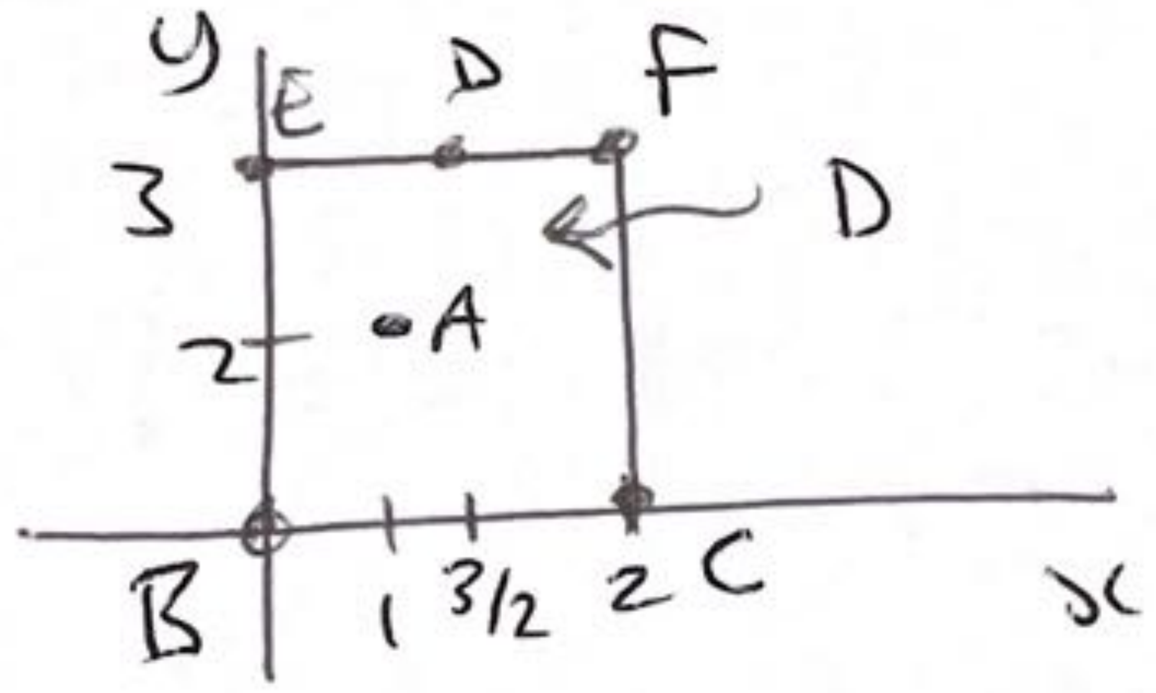
(b) Sketch the surface and the curve where $\phi = \pi/4$.

$\phi = \pi/4$ gives $z = 3\cos\pi/4 = \frac{3}{\sqrt{2}}$.

So the curve where $\phi = \pi/4$ is intersection of sphere with horizontal plane $z = 3/\sqrt{2}$, which is a circle of latitude



(5) [13 pts] Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - xy + y$ on the rectangle $[0,2] \times [0,3]$.



LABEL	(x,y)	$f(x,y)$	
A	$(1,2)$	1	
B	$(0,0)$	0	ABS min
C	$(2,0)$	4	ABS MAX
D	$(3/2, 3)$	$3/4$	
E	$(0,3)$	3	
F	$(2,3)$	1	

① CRITICAL PTS IN D :

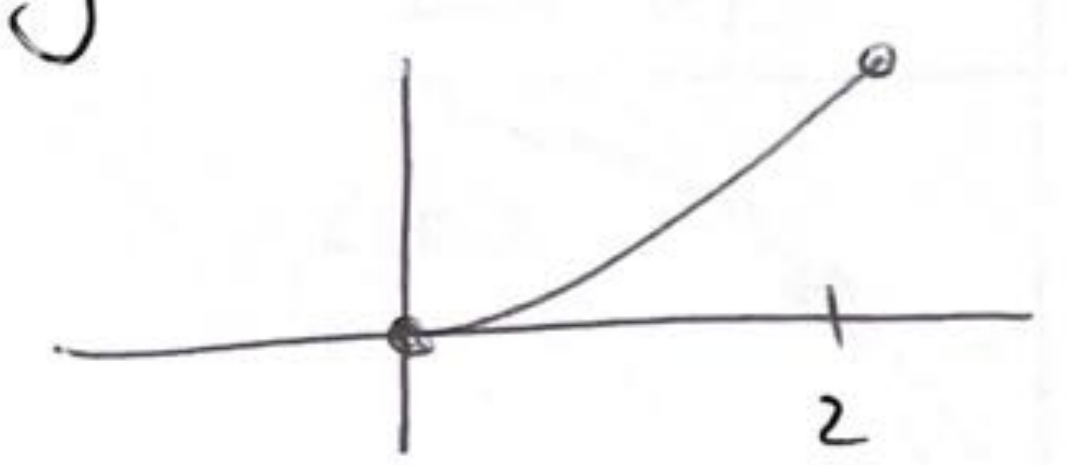
$$0 = \frac{\partial f}{\partial x} = 2x - y$$

$$0 = \frac{\partial f}{\partial y} = -x + 1 \Rightarrow x = 1 \text{ and } y = 2x = 2$$

$$(x,y) = (1,2) \quad f(1,2) = 1 \quad (A)$$

② (a) $\boxed{y = 0, \quad 0 \le x \le 2}$

$$g(x) = x^2$$



$(x,y) = (0,0) \quad f(0,0) = 0 \quad (B)$

$(x,y) = (2,0) \quad f(2,0) = 2^2 = 4 \quad (C)$

(b) $\boxed{y = 3, \quad 0 \le x \le 2}$

$$g(x) = f(x,3) = x^2 - 3x + 3$$

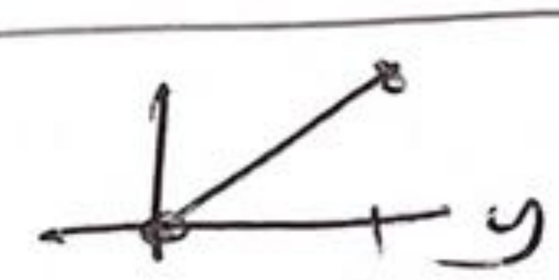
$$0 = g'(x) = 2x - 3 \Rightarrow x = \frac{3}{2}$$

$f(3/2, 3) = \frac{9}{4} - 3 \cdot \frac{3}{2} + 3 = 3/4 \quad (D)$

$f(0,3) = 3 \quad (E) \quad f(2,3) = 1 \quad (F)$

(c) $\boxed{x = 0, \quad 0 \le y \le 3}$

$g(y) = y.$

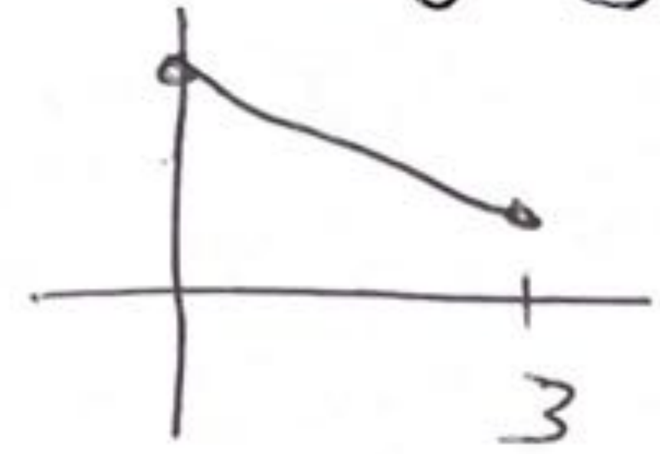


$(x,y) = (0,0) \quad f = 0. \quad (B)$

$(x,y) = (0,3) \quad f = 3 \quad (E)$

(d) $\boxed{x = 2 \quad 0 \le y \le 3}$

$$g(y) = 4 - 2y + y = 4 - y.$$



$f(2,0) = 4 \quad (C)$

$f(2,3) = 1 \quad (F)$

(6) [13 pts] Use the Method of Lagrange Multipliers to find the absolute maximum and minimum values of the function $f(x, y) = x^2y$ subject to the constraint $x^2 + y^2 = 3$.

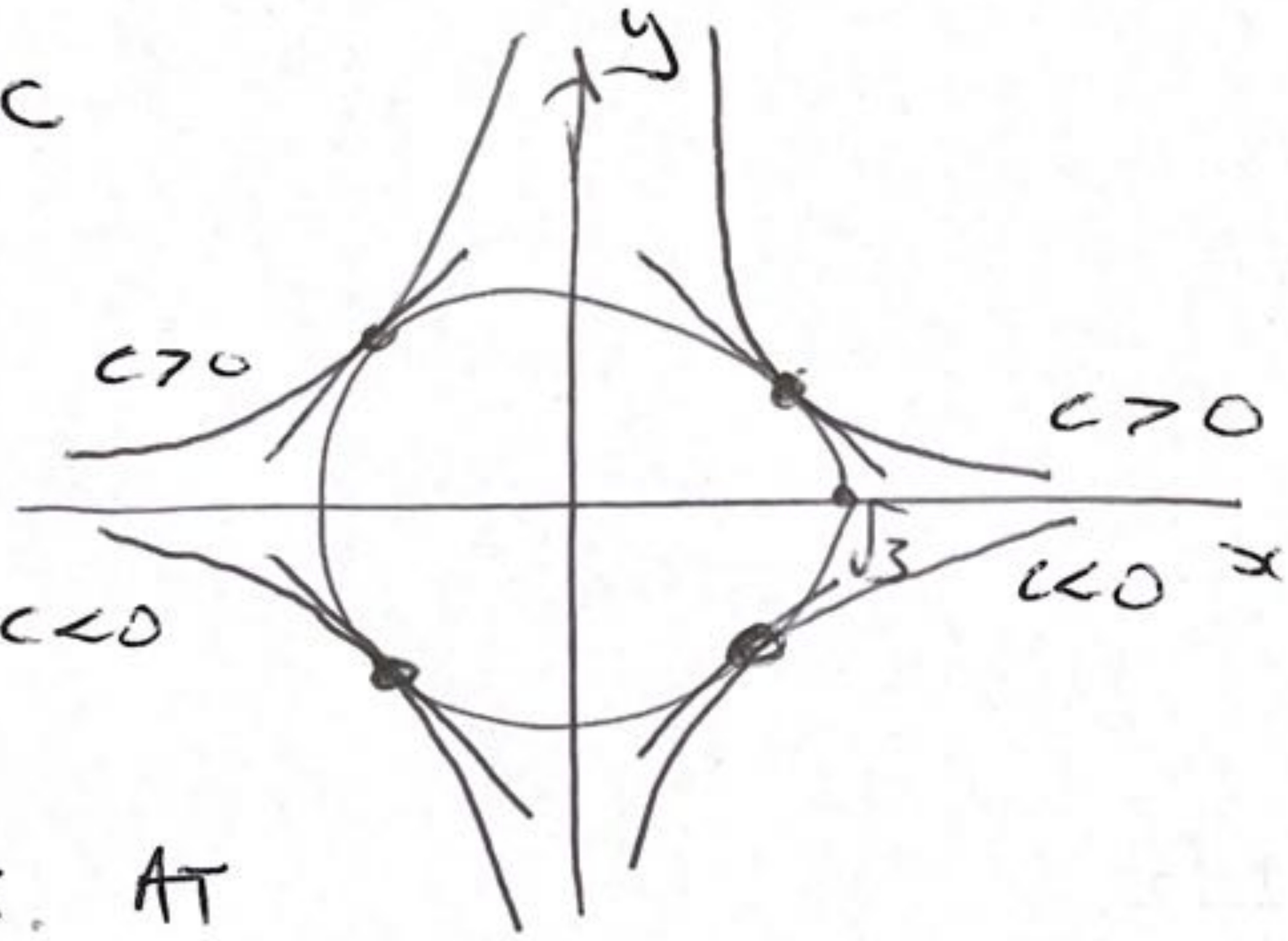
GEO METHOD

(A) $\nabla f = \vec{0}$ AT (1) $2xy = 0$, (2) $x^2 = 0$ $\Rightarrow$ $x = 0$, y ANYTHING

So get critical points of f all along y axis.
Since also need to be on constraint curve
get 2 CPTS @ $(0, \pm\sqrt{3})$.

(B) COMMON TANGENTS:

$x^2y = c$



6CPTS IN TOTAL

4 CPTS. AT

(a, b), $(a, -b)$, $(-a, b)$, $(-a, -b)$

for some a, b with $a^2 + b^2 = 3$.

PTO

$\boxed{\text{ALG METHOD}}$ $f = x^2 y$, $g = x^2 + y^2 = 3$

$f_x = \lambda g_x$: $2xy = \lambda 2x$ (1)

$f_y = \lambda g_y$: $x^2 = \lambda 2y$ (2)

$g = k$: $x^2 + y^2 = 3$ (3)

By (1) $x(y - \lambda) = 0 \Rightarrow x = 0$ or $y = \lambda$

$\boxed{x = 0}$ By (3) $y = \pm\sqrt{3}$

By (2) $\lambda = 0$.

$(x, y, \lambda) = (0, \pm\sqrt{3}, 0)$

$f(0, \pm\sqrt{3}) = 0$.

$\boxed{y = \lambda}$

By (2) $x^2 = 2\lambda^2$, $y = \lambda$

By (3) $2\lambda^2 + \lambda^2 = 3$

$3\lambda^2 = 3$

$\lambda = \pm 1$ $x = \pm\sqrt{2}$

$(x, y, \lambda) = (\sqrt{2}, 1, 1)$ $f(\sqrt{2}, 1) = 2$ ← ABS MAX

$(\sqrt{2}, -1, -1)$ $f(\sqrt{2}, -1) = -2$ ← ABS MIN

$(-\sqrt{2}, 1, 1)$ $f(-\sqrt{2}, 1) = 2$ ← ABS MAX

$(-\sqrt{2}, -1, 1)$ $f(-\sqrt{2}, -1) = -2$. ← ABS MIN

GEO + ALG METHODS AGREE!