LAST NAME:	FIRST NAME:	CIRCLE:	Khoury 5:30pm	Coskunuzer 8:30am
NEWTON	ISAAC.	Coskunuzer 11:30am	Zweck 1pm	Zweck 4pm

MATH 2415 [Fall 2023] Exam I

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. This 75 minute exam is worth 75 points. **Points will be recorded on the top of the second page.**

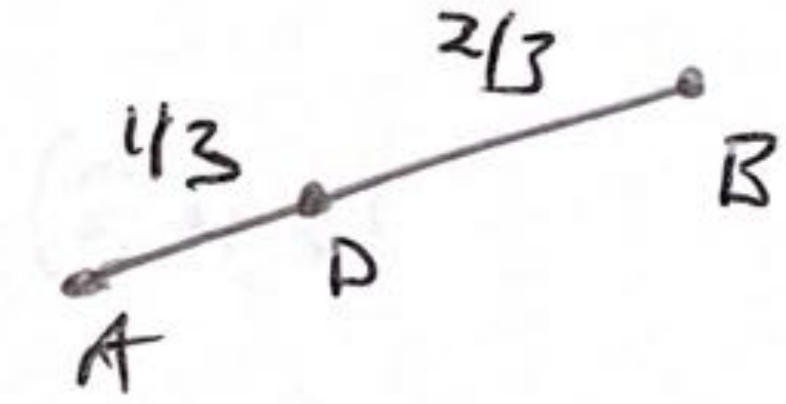
(1) [13 pts] Let $A = (1, 2, -3)$, $B = (4, 8, 0)$ and $C = (7, -1, 6)$.

(a) Find the point D on the line segment from A to B for which $2|\overrightarrow{AD}| = |\overrightarrow{DB}|$. **Hint:** Parametrize the line segment $\overrightarrow{AB}$.

$\vec{p} = A = (1, 2, -3)$ $\quad \vec{v} = \overrightarrow{AB} = B - A = (4, 8, 0) - (1, 2, -3)$

$= (3, 6, 3)$

$\vec{r}(t) = \vec{p} + t\vec{v} = (1, 2, -3) + t\,(3, 6, 3)$

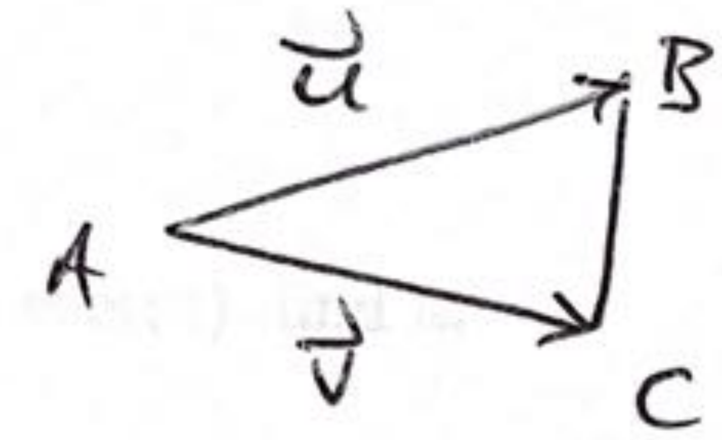


$D = \vec{r}(1/3) = (1, 2, -3) + \frac{1}{3}(3, 6, 3)$

$= (1, 2, -3) + (1, 2, 1)$

$\boxed{D = (2, 4, -2)}$

(b) Find the area of the triangle ABC.



$\vec{u} = \overrightarrow{AB} = (3, 6, 3)$

$\vec{v} = \overrightarrow{AC} = (7, -1, 6) - (1, 2, -3)$

$= (6, -3, 9)$

$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 6 & 3 \\ 6 & -3 & 9 \end{vmatrix}$

$= 63\vec{i} - 9\vec{j} + -45\vec{k}$

AREA $= \frac{1}{2}|\vec{u} \times \vec{v}|$

$= \frac{1}{2}\sqrt{63^2 + 9^2 + 45^2}$

$= \frac{9}{2}\sqrt{7^2 + 1 + 5^2} = \frac{9}{2}\sqrt{75} = \frac{45}{2}\sqrt{3}$

1 /13	2 /12	3 /13	4 /12	5 /13	6 /12	T /75

(2) [12 pts] Let $\mathbf{u} = \langle 3, -2, 1 \rangle$ and $\mathbf{v} = \langle 5, 4, 3 \rangle$.

(a) Find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$.

$$\mathrm{COMP}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} = \frac{(3,-2,1) \cdot (5,4,3)}{\sqrt{9+4+1}}$$

$$= \frac{15-8+3}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

(b) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\mathrm{PROJ}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} = \frac{10}{\left(\sqrt{25+16+9}\right)^2} (5, 4, 3)$$

$$= \frac{1}{5}(5,4,3) = \left(1, \frac{4}{5}, \frac{3}{5}\right)$$

(c) Let $\mathbf{w} = \langle 1, 7, a \rangle$. If the vector projection of $\mathbf{w}$ onto $\mathbf{v}$ is $\mathbf{0}$ (the zero vector), find a.

$$\mathrm{PROJ}_{\vec{v}}(\vec{w}) = \frac{\vec{w} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} = \vec{0}$$

holds exactly when $\vec{v} \cdot \vec{w} = 0$

NOW

$$0 = \vec{v} \cdot \vec{w} = (5,4,3) \cdot (1,7,a) = 5 + 28 + 3a$$

$$3a + 33 = 0$$

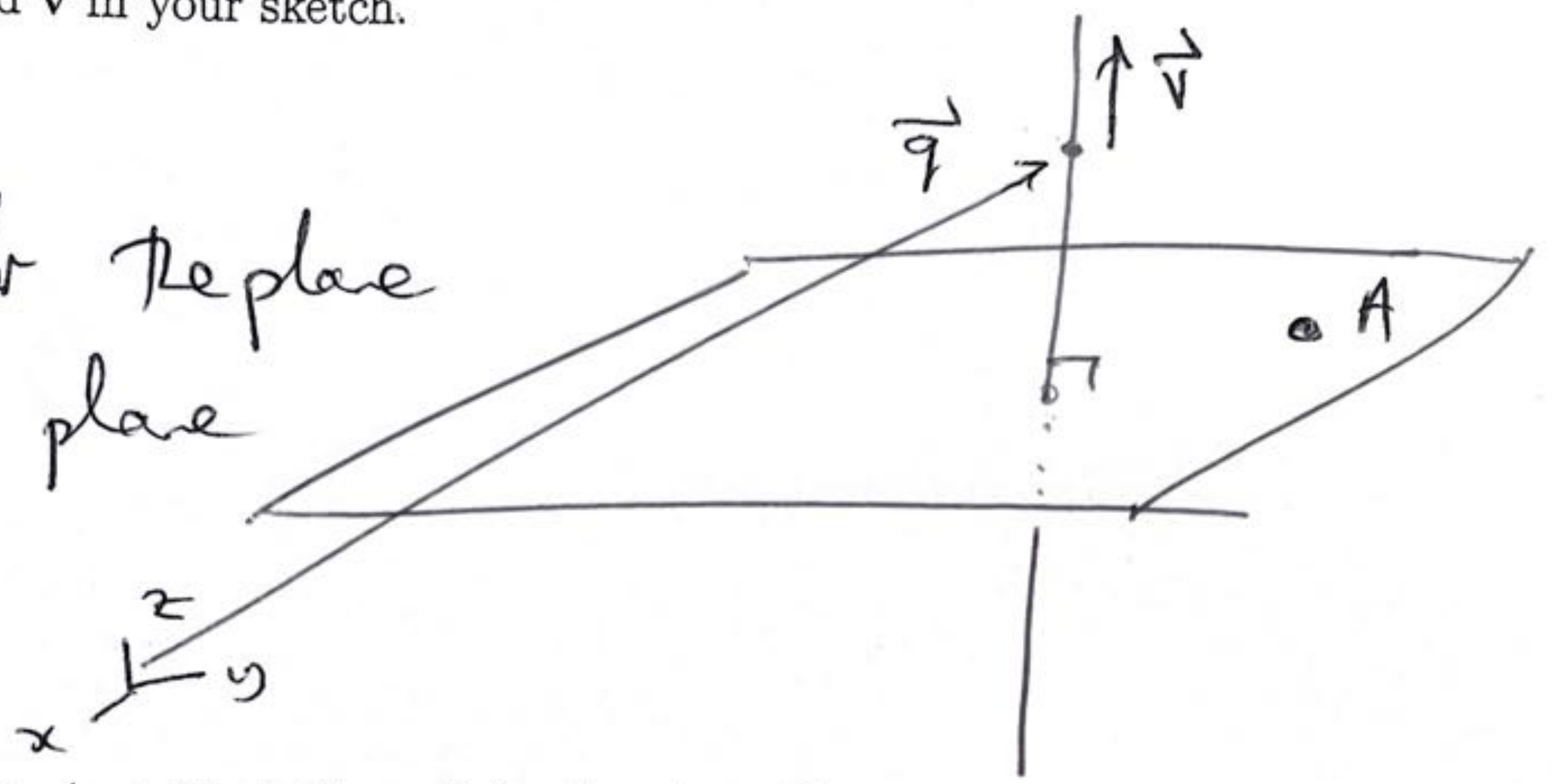
$$\boxed{a = -11}$$

(3) [13 pts] Let $\mathcal{P}$ be the plane through the point $A = (2, 1, 0)$ that is perpendicular to the line with parameterization $\mathbf{r}(t) = \mathbf{q} + t\mathbf{v} = (2 + 5t)\mathbf{i} + (2 - t)\mathbf{j} + 4t\mathbf{k}$.

(a) Draw a schematic diagram showing the relationship between the plane and the line. Include the point, A, and the vectors, $\mathbf{q}$ and $\mathbf{v}$ in your sketch.

$\vec{v}$ is a normal for the plane

A is a point on plane



(b) Find an equation of the form $Ax + By + Cz = D$ for the plane, $\mathcal{P}$.

$$0 = (\vec{r} - \vec{p}) \cdot \vec{n}$$

$$\vec{p} = A = (2, 1, 0)$$
$$\vec{n} = \vec{v} = (5, -1, 4)$$
$$\vec{r} = (x, y, z)$$

$$0 = ((x-2), y-1, z-0) \cdot (5, -1, 4)$$

gives $\boxed{5x - y + 4z = 9}$

(c) Find a parameterization of the plane, $\mathcal{P}$.

THERE ARE ∞ # CORRECT ANSWERS HERE ARE TWO

<u>METHOD I</u> SET $x = s$

$y = t$

$$z = \frac{9 - 5x + y}{4} = \frac{9 - 5s + t}{4}$$

$$\vec{r}(s,t) = \left(s, t, \frac{9 - 5s + t}{4}\right)$$

<u>METHOD II</u> $\vec{r}(s,t) = \vec{p} + s\vec{v} + t\vec{w}$

$\vec{v} = \vec{AB} = (\frac{9}{5} - 2, -1, 0) = (-\frac{1}{5}, -1, 0)$

$\vec{w} = \vec{AC} = (-2, -10, 0)$

3 POINTS IN PLANE:

$A = (2, 1, 0)$

$B = (\frac{9}{5}, 0, 0)$

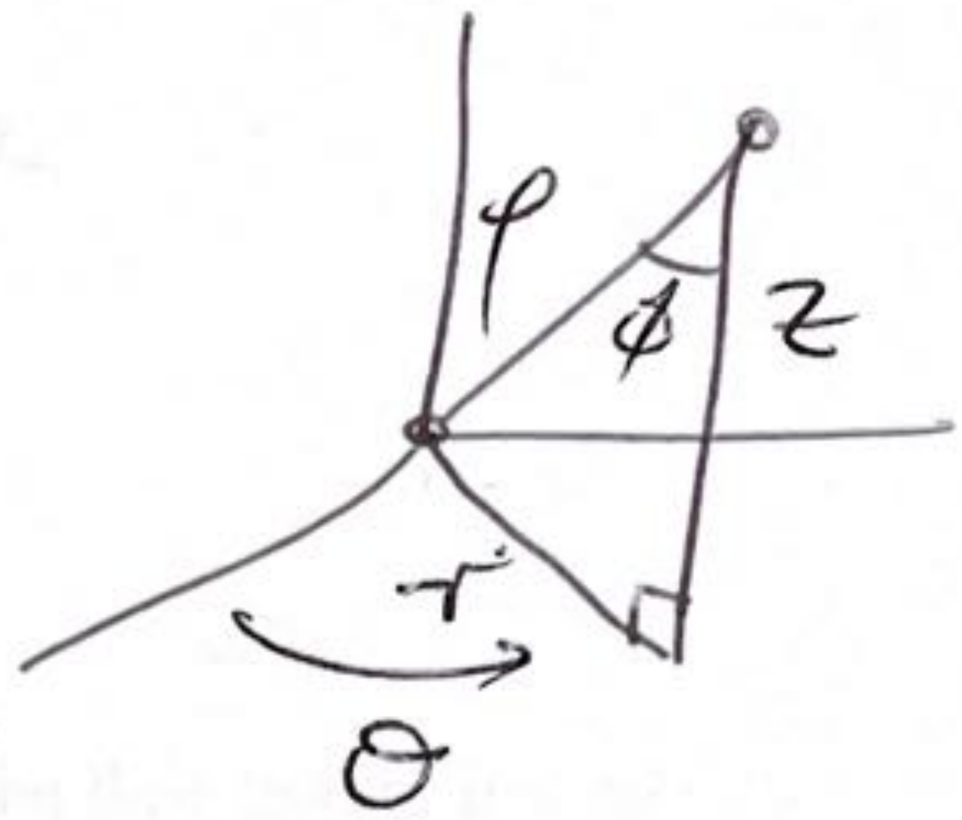
$C = (0, -9, 0)$

$\vec{r}(s,t) = (2, 1, 0) + s(-\frac{1}{5}, -1, 0) + t(-2, -10, 0)$

CHECK: $\vec{v} \perp \vec{n}$

(4) [12 pts]
(a) Let P be the point with spherical coordinates $(\rho, \theta, \phi) = (2, \pi/3, \pi/4)$.

(i) Find the cylindrical coordinates of P.



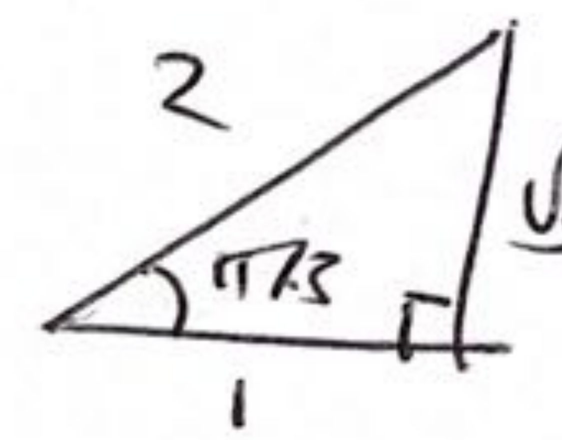
$$r = \rho \sin\phi = 2 \sin \pi/4 = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\theta = \theta = \pi/3$$

$$z = \rho\cos\phi = 2\cos\pi/4 = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\boxed{(r, \theta, z) = (\sqrt{2}, \pi/3, \sqrt{2})}$$

(ii) Find the rectangular coordinates of P.



$$x = r\cos\theta = \frac{2}{\sqrt{2}} \cos \pi/3 = \frac{1}{\sqrt{2}}$$

$$y = r\sin\theta = \frac{2}{\sqrt{2}} \sin \pi/3 = \sqrt{2}\,\frac{\sqrt{3}}{2} = \sqrt{\frac{3}{2}}$$

$$z = z = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$(x, y, z) = \left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}, \sqrt{2}\right)$$

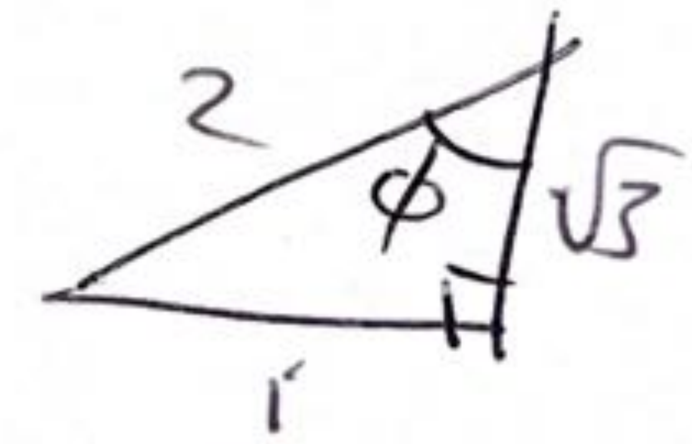
(b) Convert the equation $z = \sqrt{3x^2 + 3y^2}$ into an equation involving spherical coordinates ρ, θ and ϕ.

$$\left.\begin{array}{l} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \end{array}\right] \Rightarrow x^2 + y^2 = \rho^2 \sin^2\phi\,(\cos^2\theta + \sin^2\theta)$$

$$= \rho^2\sin^2\phi$$

$$z = \rho\cos\phi$$

So $z = \sqrt{3}\sqrt{x^2+y^2}$



is $\cancel{\rho\cos\sqrt{}}$ $\rho\cos\phi = \sqrt{3}\,\rho\sin\phi$

$$\tan\phi = \frac{1}{\sqrt{3}} \quad \text{So } \boxed{\phi = \pi/6}$$

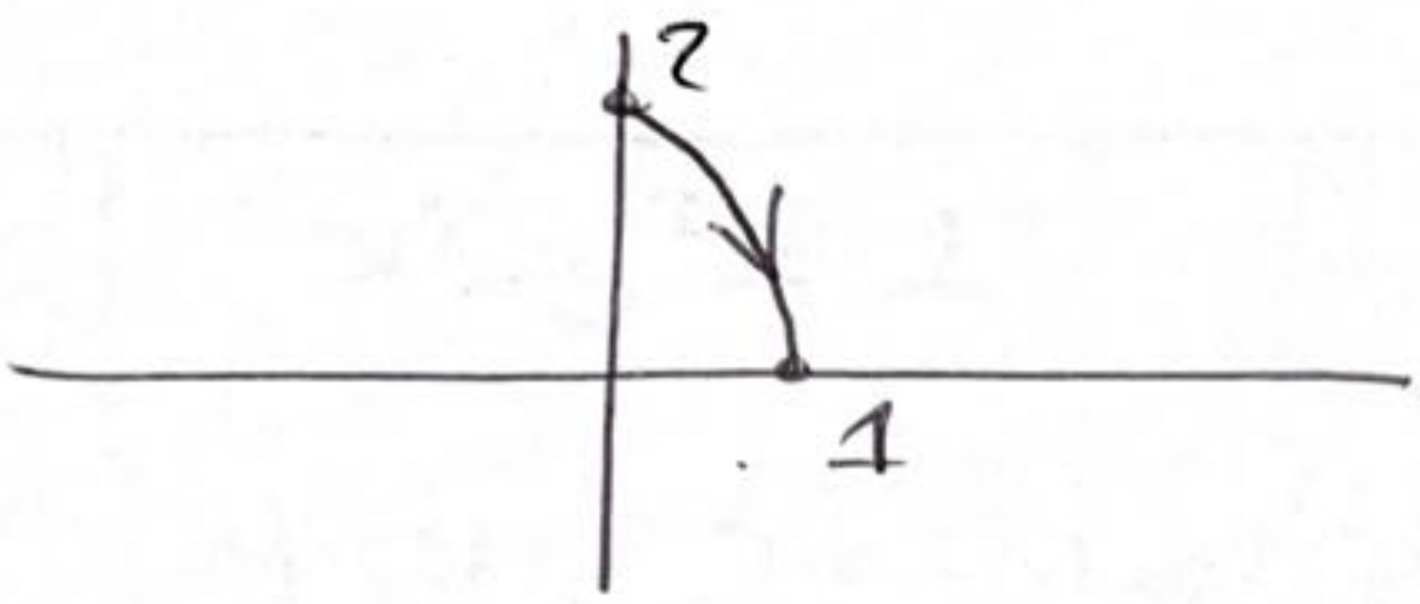
(5) [13 pts] Let C be the parametrized curve given by $(x, y) = \mathbf{r}(t) = (\sin t, 2\cos t)$ for $0 \le t \le \pi/2$.
(a) Eliminate t to obtain an equation relating x and y.

$x = \sin t, \quad y = 2\cos t$

$1 = \cos^2 t + \sin^2 t = \left(\frac{y}{2}\right)^2 + x^2$

$$\boxed{x^2 + \left(\frac{y}{2}\right)^2 = 1}$$

(b) Sketch the curve, clearly marking the start and end points and the direction of motion.



$\vec{r}(0) = (0, 2)$

$\vec{r}(\pi/2) = (1, 0)$

1/4-ELLIPSE

(c) Find a parametrization for the tangent line to the curve C at the point where $t = \pi/4$.

$\vec{P} = \vec{r}(\pi/4) = (\cos(\pi/4), 2\sin \pi/4) = \left(\frac{1}{\sqrt{2}}, 2\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, 2)$

$\vec{v} = \vec{r}\,'(\pi/4), \quad \vec{r}\,'(t) = (\cos t, -2\sin t)$

$\vec{v} = \left(\frac{1}{\sqrt{2}}, -2\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, -2)$

THERE ARE ∞ # CORRECT ANSWERS. THIS IS ONE

$\vec{\ell}(t) = \vec{P} + (t - \pi/4)\vec{v} = \boxed{\frac{1}{\sqrt{2}}(1, 2) + (t - \pi/4)\frac{1}{\sqrt{2}}(1, -2)}$

(d) Find constants a and b and a function $f(t)$ so that the length of C is $L = \int_a^b f(t)\,dt$, where C is the curve whose parametrization, $(x, y) = \mathbf{r}(t)$, is given above.

$L = \int_{t=a}^{t=b} |\vec{r}\,'(t)|\,dt \qquad a = 0, \; b = \pi/2.$

$f(t) = |\vec{r}\,'(t)| = \sqrt{(\cos t)^2 + (-2\sin t)^2}$

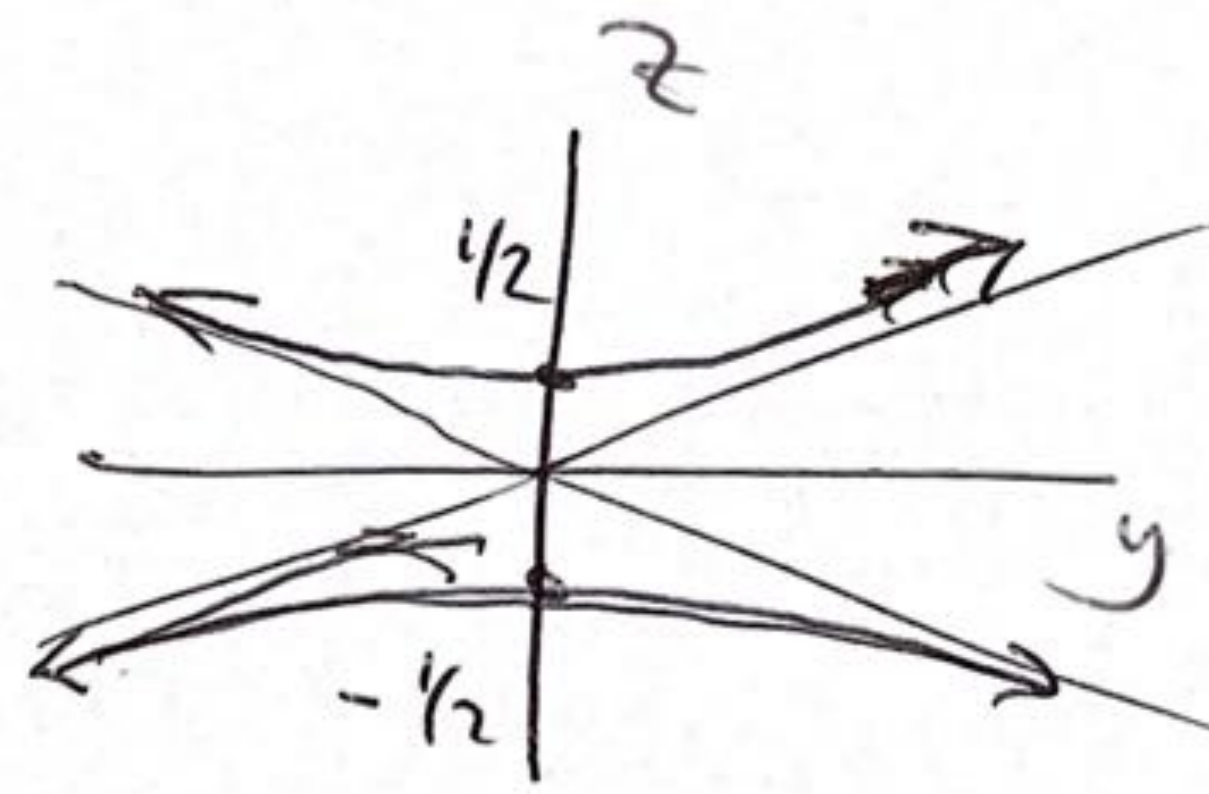
$$\boxed{\begin{aligned} &f(t) = \sqrt{(\cos t)^2 + 4(\sin t)^2} \\ \text{OR} \quad &f(t) = \sqrt{1 + 3\sin^2 t} \end{aligned}}$$

(6) [12 pts] Make labelled sketches of the traces (slices) of the surface $x^2 - y^2 + 4z^2 = 1$ in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Be sure to include any asymptotes and intercepts in your sketches. Then make a labelled sketch of the surface.

$\boxed{x=0}$ $\quad 4z^2 - y^2 = 1$

INT $\quad y=0,\ z = \pm\frac{1}{2} \Rightarrow (0, \pm\frac{1}{2})$

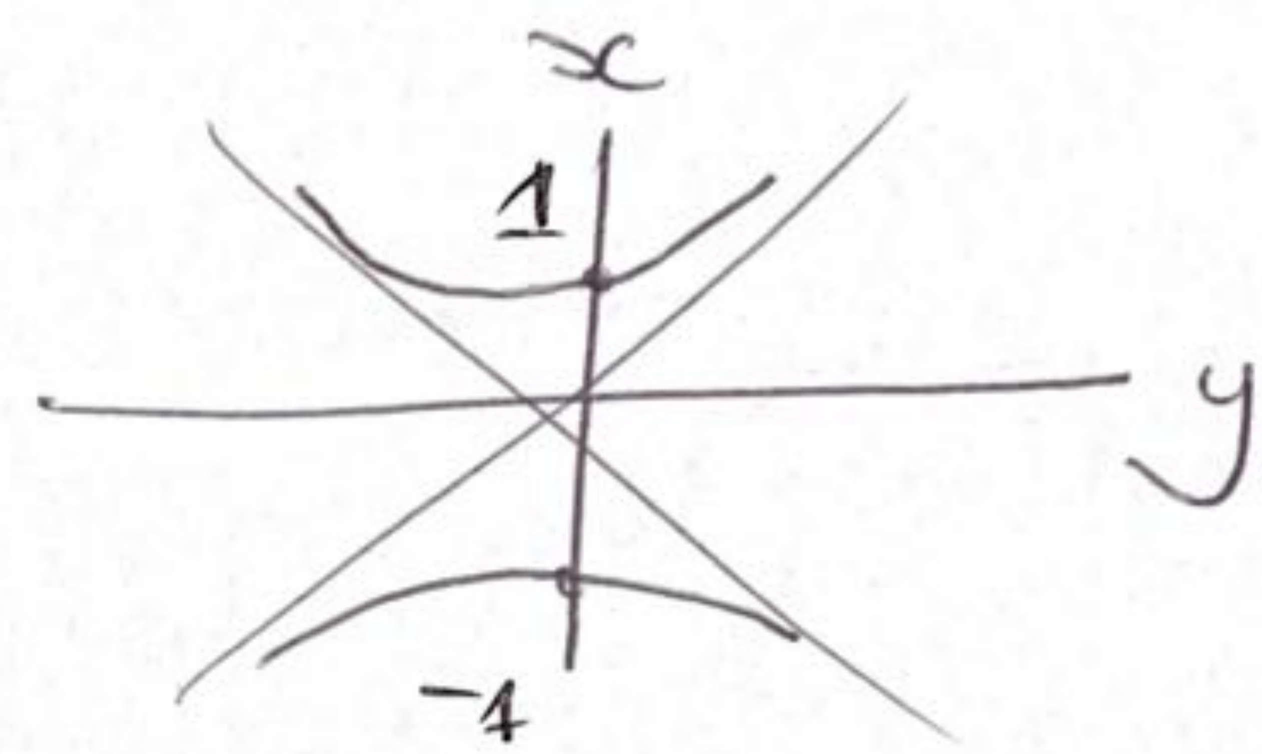
ASY $\quad 4z^2 - y^2 = 0 \Rightarrow z = \pm\frac{1}{2}y$



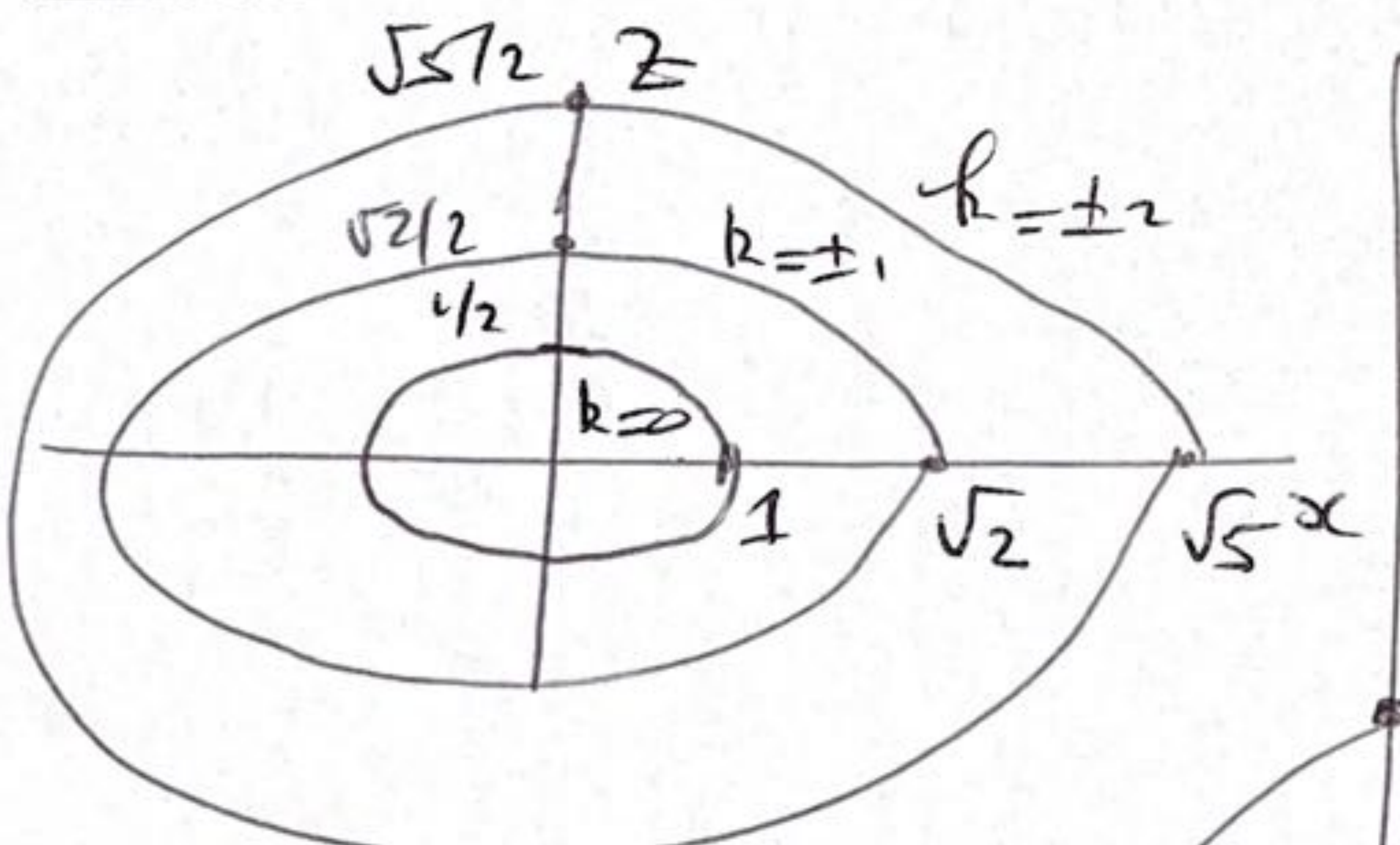
$\boxed{z=0}$ $\quad x^2 - y^2 = 1$

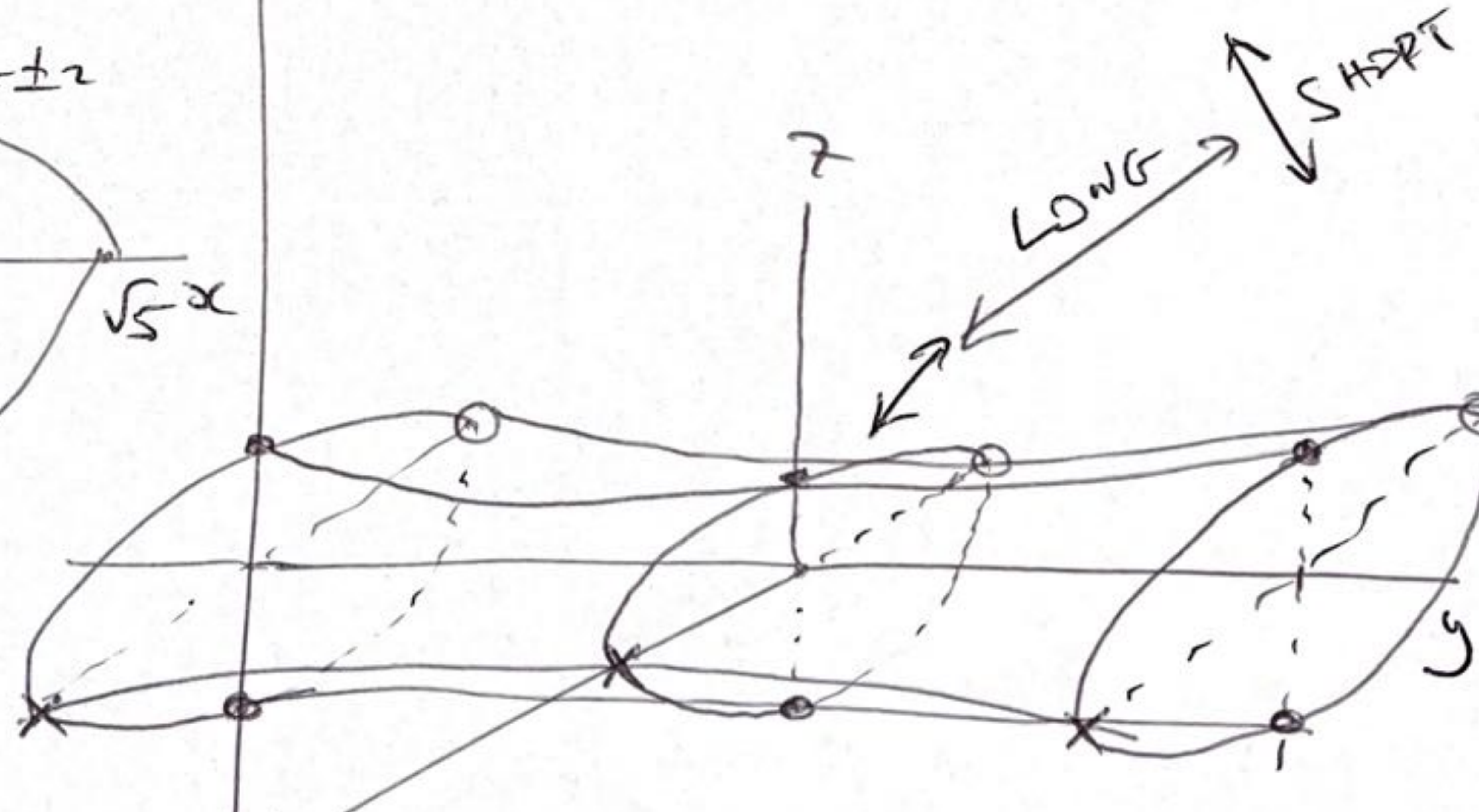
INT $\quad y=0,\ x = \pm 1 \Rightarrow (\pm 1, 0)$

ASY $\quad y = \pm x$



$\boxed{y=k}$ $\quad x^2 + 4z^2 = 1 + k^2$. ELLIPSES





ELLIPTICAL HYPERBOLOID OF 1 SHEET ALIGNED WITH y-AXIS.