LAST NAME:	FIRST NAME:	CIRCLE:	Coskunuzer 8:30am	Dahal 11:30am
LAGRANGE	JOSEPH – LOUIS	Dahal 5:30pm	Zweck 1pm	Zweck 4pm

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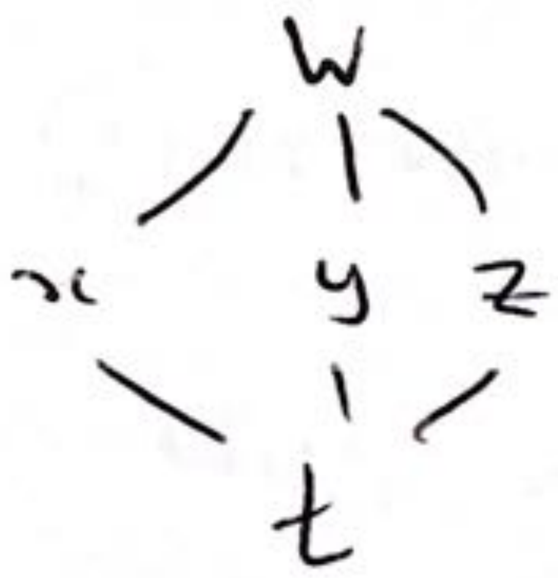
MATH 2415 [Fall 2022] Exam II, Nov 4th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points. **Your points for each problem will be recorded on the top of the second page.**

(1) [12 pts]

(a) Suppose that $w = f(x, y, z)$, where $x = x(t)$, $y = y(t)$ and $z = z(t)$. Use a tree diagram to write out a formula for $\frac{dw}{dt}$. Use this formula to find $\frac{dw}{dt}$ at $t = 0$ when $f(x, y, z) = e^{x^2+y^2+z}$, $x = t^3$, $y = \sin t$, $z = 3t$.



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \left(2x(t) \cdot 3t^2 + 2y(t)\cos t + 1.3\right) e^{x^2(t) + y^2(t) + z(t)}$$

$$\frac{dw}{dt}(0) = (0 + 0 + 3)\, e^{0+0+0} = 3.$$

(b) Find the equation for the tangent plane to the graph of $z = f(x, y) = y^2 e^x$ at $(0, 1)$. Use this equation to approximate $f(0.2, 1.1)$.

$$\frac{\partial f}{\partial x} = y^2 e^x = 1 \text{ @ } (0,1)$$

$$\frac{\partial f}{\partial y} = 2y e^x = 2 \text{ @ } (0,1)$$

$$f(0,1) = 1$$

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$z = 1 + 1(x - 0) + 2(y - 1) = x + 2y - 1$$

$$f(0.2, \cancel{1}1.1) \simeq 0.2 + 2(1.1) - 1 = 1.4$$

1 /12	2 /10	3 /14	4 /15	5 /12	6 /12	T /75

(2) [10 pts]
(a) Show that $u(x, y) = \ln(x^2 + y^2)$ satisfies the partial differential equation $u_{xx} + u_{yy} = 0$.

$$u_x = \frac{2x}{x^2+y^2}$$

$$u_{xx} = \frac{2(x^2+y^2) - 2x\,\overset{2x}{\cancel{(x^2+y^2)}}}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

SWAPPING roles of x, y :

$$u_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2} = -u_{xx}$$

So $u_{xx} + u_{yy} = 0$.

(b) Let $f(x, y) = xy$. As x_0 increases from $x_0 = 1$, does the slope of the slice of the graph of f in the plane $x = x_0$ increase or decrease? Why?

$$\frac{\partial^2 f}{\partial x \partial y} = 1 > 0.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$ = Rate of Change wrt x of Slope of Slice of Graph in planes $x =$ constant.

So answer is <u>INCREASES</u>.

(3) [14 pts] Let $f(x, y) = ye^{2x}$.
(a) Find the gradient of f at the point $(0, 1)$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2y\, e^{2x}, e^{2x}\right)$$
$$= (2, 1) \quad @ \ (0, 1).$$

(b) Find the directional derivative of f at the point $(0, 1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$$\vec{x}_0 = (0, 1) \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}(1, 1)$$

$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u} = \nabla f(0, 1) \cdot \vec{u}$$
$$= (2, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$$

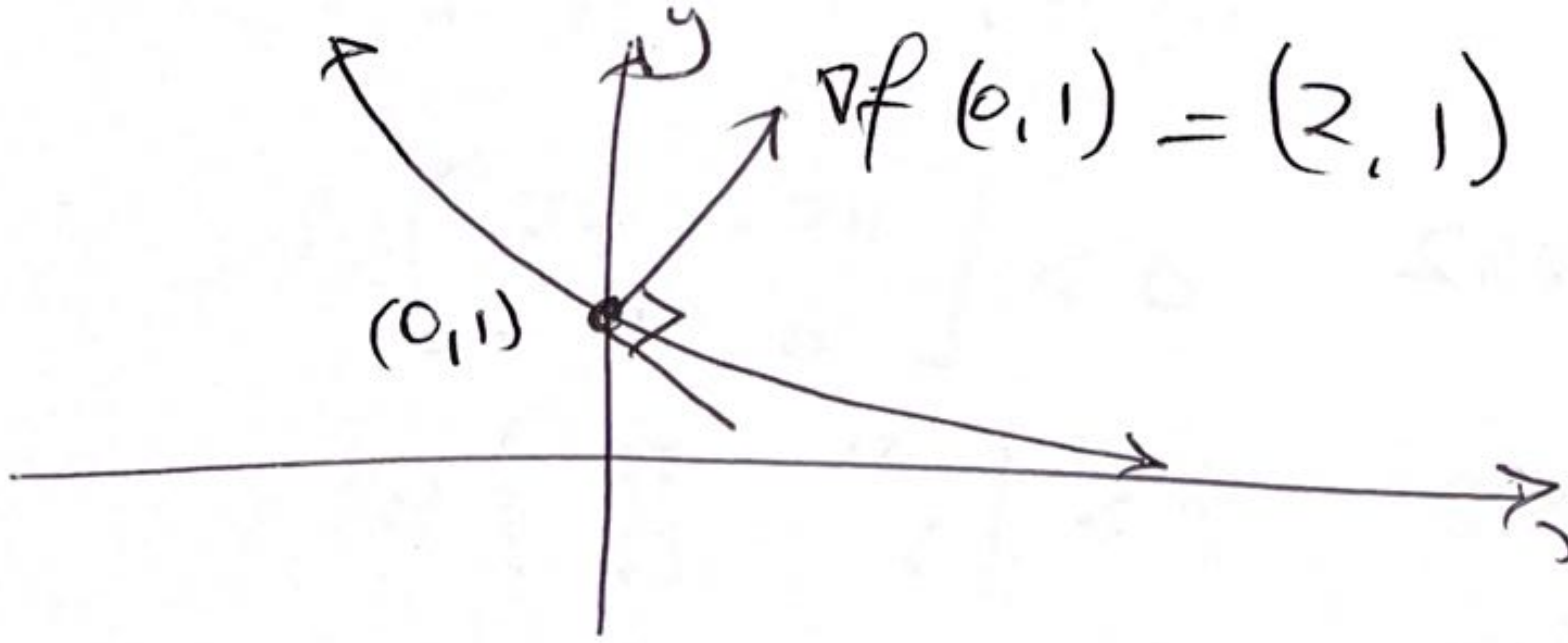
(c) Find the direction of the minimum rate of change in f at (0,1). Also find the minimum rate of change.

$$\vec{u} = \frac{-\nabla f(0, 1)}{|\nabla f(0, 1)|} = -\frac{1}{\sqrt{5}}(2, 1) \text{ is Direction}$$

$$\text{Min Rate of Change} = -|\nabla f(0, 1)| = -\sqrt{5}$$

(d) Sketch the level curve $f(x, y) = 1$. Add the vector $\nabla f(0, 1)$ to your sketch.

$$y e^{2x} = 1 \Rightarrow y = e^{-2x}$$



(4) [15pts] Find and classify all critical points of the function $f(x, y) = 2x^3 - 3x^2y + 3y^2 + 12x^2$.

$$0 = \frac{\partial f}{\partial x} = 6x^2 - 6xy + 24x = 6x(x - y + 4) \quad ①$$

$$0 = \frac{\partial f}{\partial y} = -3x^2 + 6y \quad ②$$

By ① $x = 0$ or $y = x + 4$.

$\boxed{x = 0}$ By ② $y = 0$ Get CPT $(0, 0)$

$\boxed{y = x + 4}$ By ② $0 = -3x^2 + 6x + 24$

$= -3(x - 4)(x + 2)$

So CPTS $(4, 8)$, $(-2, 2)$

$$D = \det \begin{bmatrix} 12x - 6y + 24 & -6x \\ -6x & 6 \end{bmatrix}$$

ⓐ $D(0,0) = \det \begin{bmatrix} 24 & 0 \\ 0 & 6 \end{bmatrix} > 0$ $f_{xx}(0,0) = 24 > 0$ LOCAL MIN AT $(0,0)$

ⓑ $D(4,8) = \det \begin{bmatrix} 24 & -24 \\ -24 & 6 \end{bmatrix} < 0$ SADDLE AT $(4,8)$

ⓒ $D(-2,2) = \det \begin{bmatrix} -12 & 12 \\ 12 & 6 \end{bmatrix} < 0$ SADDLE AT $(-2, 2)$

(5) [12 pts] Let S be the surface with parametrization

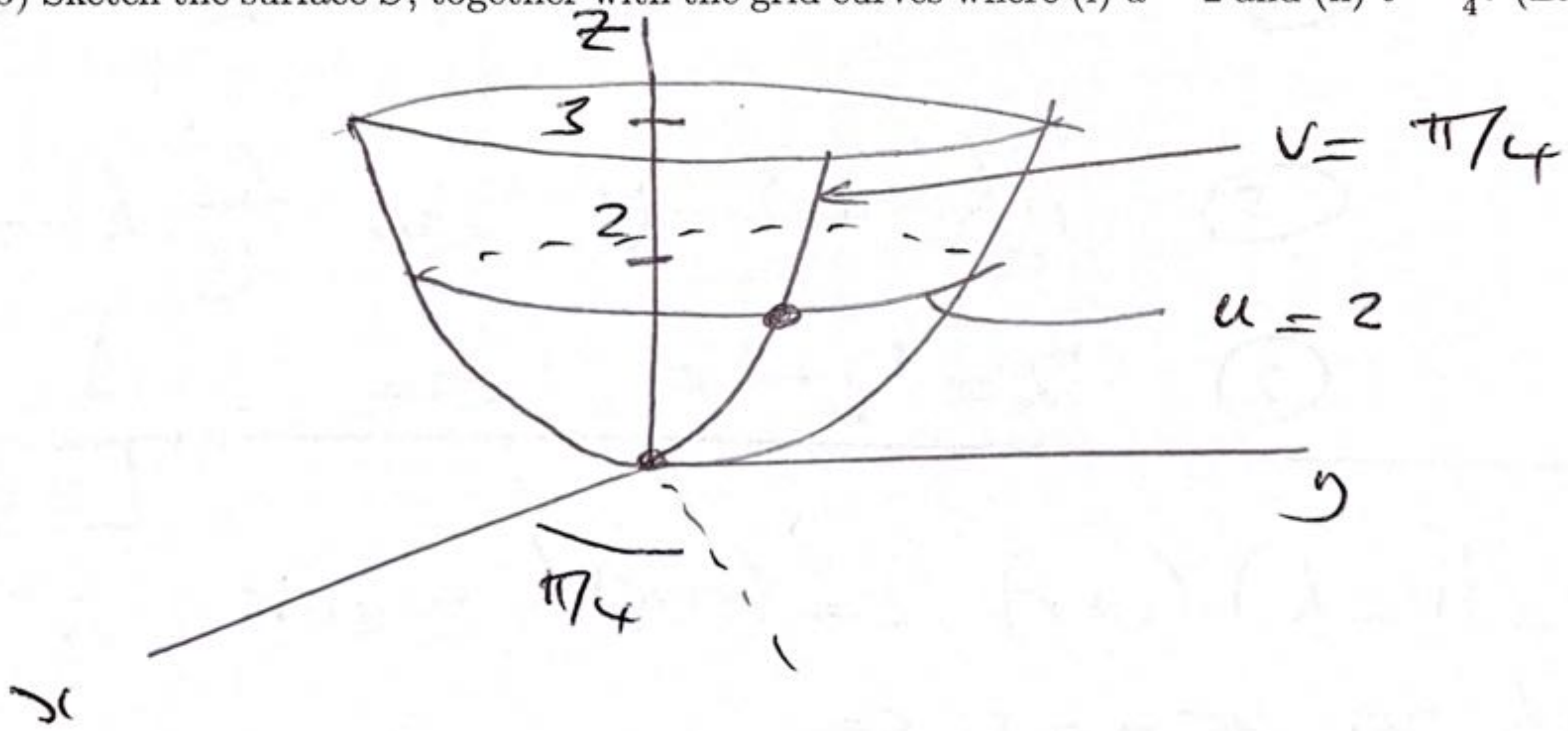
$$(x, y, z) = \mathbf{r}(u, v) = (u\cos v, u\sin v, u^2), \qquad \text{for } 0 \le u \le 3 \text{ and } 0 \le v \le 2\pi.$$

(a) Show that S is part of a paraboloid. **Hint:** Find an equation of the form $F(x, y, z) = 0$ for this surface.

$$x^2+y^2 = (u\cos v)^2 + (u\sin v)^2$$
$$= u^2(\cos^2 v + \sin^2 v)$$
$$= u^2 = z$$

So $0 = F(x,y,z) = z - x^2 - y^2$. Paraboloid

(b) Sketch the surface S, together with the grid curves where (i) $u = 2$ and (ii) $v = \frac{\pi}{4}$. (Label these curves!)

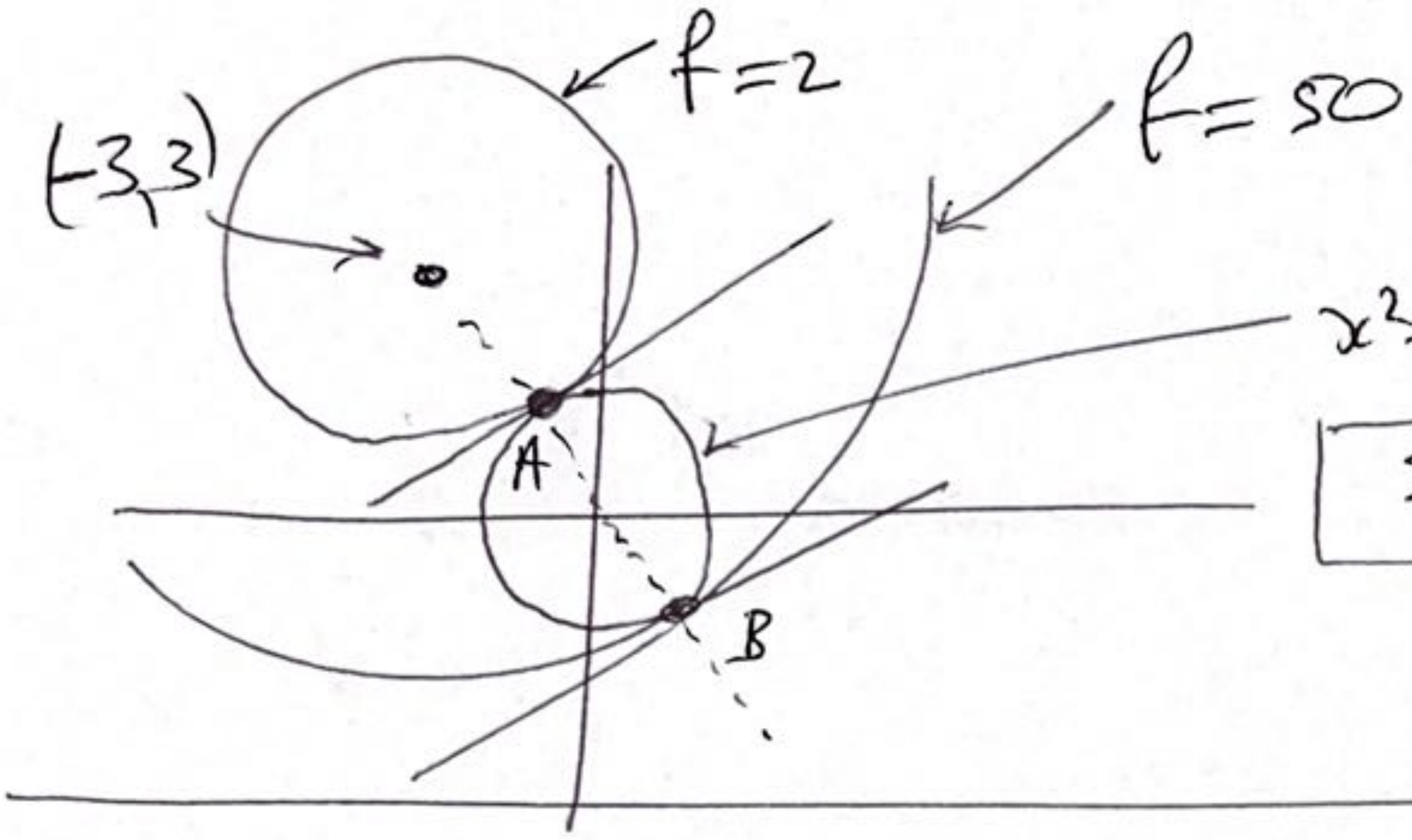


(c) Calculate a tangent vector to the grid curve where $v = \frac{\pi}{4}$ at the point $\mathbf{r}(2, \frac{\pi}{4})$.

$$\vec{V} = \frac{\partial \vec{r}}{\partial u}(2, \pi/4)$$ [on $v = \pi/4$, v is constant and u changes]

$$\frac{\partial \vec{r}}{\partial u} = (\cos v, \sin v, 2u)$$
$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 4\right) \quad @ (u.v) = (2, \pi/4)$$

(6) [12 pts] Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function $f(x,y) = (x+3)^2 + (y-3)^2$ on the circle $x^2 + y^2 = 8$.



ALSO $\nabla f = (0,0)$ ONLY AT $(-3,3)$ WHICH IS NOT ON $x^2+y^2=8$

$\boxed{2 \text{ CPTS AT } A, B}$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow x+3 = \lambda x \quad (1)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow y-3 = \lambda y \quad (2)$$

$$g = k \Rightarrow x^2+y^2 = 8 \quad (3)$$

$\boxed{\text{METHOD 1}}$

(1)+(2): $x+y = \lambda(x+y) \Rightarrow (x+y)(\lambda - 1) = 0.$

$\Rightarrow y = -x$ or $\lambda = 1$

$\boxed{\lambda = 1}$ By (1) $x+3 = x \Rightarrow 3 = 0$ DEAD END

$\boxed{y=-x}$ By (3) $2x^2 = 8 \Rightarrow x = \pm 2 \Rightarrow y = -x = \mp 2$

$(2,-2)$: $\lambda = \frac{x+3}{x} = \frac{5}{2}$ $\quad f(2,-2) = 50$

$(-2,2)$: $\lambda = \frac{x+3}{x} = -\frac{1}{2}$ $\quad f(-2,2) = 50$

$\boxed{\text{METHOD 2}}$ By (1) $x = \frac{3}{\lambda - 1}$ By (2) $y = \frac{-3}{\lambda - 1}$

Plug into (3): $\frac{18}{(\lambda-1)^2} = 8 \Rightarrow (\lambda - 1)^2 = \frac{9}{4}$

$\Rightarrow \lambda - 1 = \pm \frac{3}{2} \Rightarrow \lambda = -\frac{1}{2}$ OR $\lambda = 5/2$ ETC