| LAST NAME: | FIRST NAME: | CIRCLE: | Coskunuzer 8:30am | Dahal 11:30am |
|---|---|---|---|---|
| EULER | LEONHARD | Dahal 5:30pm | Zweck 1pm | Zweck 4pm |

| 1 /12 | 2 /12 | 3 /12 | 4 /15 | 5 /12 | 6 /12 | T /75 |
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# MATH 2415 [Fall 2022] Exam I, Sep 30th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [12 pts] Let $\mathbf{u} = 4\mathbf{i} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(a) Find the scalar projection of **u** onto **v**.

$$\text{COMP}_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{(4,0,3) \cdot (2,-1,-2)}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}$$

(b) Find the vector projection of **v** onto **u**.

$$\text{PROJ}_{\vec{u}}\, \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \frac{\vec{u}}{|\vec{u}|} = \frac{2}{5^2} (4,0,3)$$

$$= \frac{2}{25} (4,0,3)$$

(c) Find the angle between **u** and **v**. [Your answer should be in terms of an inverse trigonometric function.]

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{2}{5 \times 3} = \frac{2}{15}$$

$$\theta = \arccos\left(\frac{2}{15}\right)$$

(2) [12 pts] Let $\mathbf{u} = (3, 0, -2)$ and $\mathbf{v} = (-4, 1, 2)$.

(a) Find a vector $\mathbf{w}$ that has length one and is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -2 \\ -4 & 1 & 2 \end{vmatrix} = 2\vec{i} + 2\vec{j} + 3\vec{k}$$

$$|\vec{u} \times \vec{v}| = \sqrt{4+4+9} = \sqrt{17}$$

$$\vec{w} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \frac{1}{\sqrt{17}}(2, 2, 3)$$ or its negative

(b) Find the volume of the parallelepiped generated by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{k} = (0, 0, 1)$.

$$\text{VOL} = |(\vec{u} \times \vec{v}) \cdot \vec{k}| = |(2\vec{i} + 2\vec{j} + 3\vec{k}) \cdot \vec{k}|$$

$$= |3| = 3$$

(a) Find a parametrization of the line tangent to the curve, $C$, when $t = \frac{\pi}{4}$.

$$\vec{l}(s) = \vec{p} + (s - \pi/4)\vec{v} \qquad \vec{p} = \vec{r}(\pi/4) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ln\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$\vec{v} = \vec{r}\,'(\pi/4)$$

$$\vec{r}\,'(t) = \left(-\sin t, \cos t, \frac{-\sin t}{\cos t}\right)$$

$$\vec{v} = \vec{r}\,'(\pi/4) = \left(-1/\sqrt{2}, 1/\sqrt{2}, -1\right)$$

$$\vec{l}(s) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ln\left(\frac{1}{\sqrt{2}}\right)\right) + (s - \pi/4)\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right)$$

(b) Show that the length of the segment of the curve, $C$, from $t = 0$ to $t = \frac{\pi}{4}$ is $L = \int_0^{\pi/4} \sec t\, dt$.

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{\pi/4} \sqrt{(-\sin t)^2 + (\cos^2 t) + (-\tan t)^2}\, dt$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 t}\, dt$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 t}\, dt \qquad \cos^2 t + \sin^2 t = 1 \Rightarrow 1 + \tan^2 t = \sec^2 t$$

$$= \int_0^{\pi/4} \sec t\, dt \quad \text{as } \cos t > 0 \text{ for } 0 < t < \pi/4$$

(4) [15pts]

(a) Parametrize the curve of intersection of the surfaces $x = y^2 - z^2$ and $y^2 + z^2 = 9$.

$$\left.\begin{aligned} y &= 3\cos t \\ z &= 3\sin t \end{aligned}\right] \Rightarrow y^2+z^2 = 9$$

$$x = y^2 - z^2 = 9(\cos^2 t - \sin^2 t)$$

So $\vec{r}(t) = (9(\cos^2 t - \sin^2 t), 3\cos t, 3\sin t)$

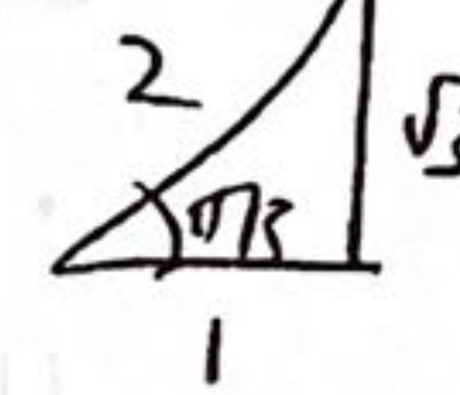
for $0 \le t \le 2\pi$

(b) Let $P$ be the point with spherical coordinates $(\rho, \theta, \phi) = (4, -\frac{\pi}{4}, \frac{\pi}{3})$. Find the rectangular coordinates of $P$.

$$x = \rho \sin\phi \cos\theta = 4 \sin \pi/3 \cos(-\pi/4) = 4 \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} = \sqrt{6}$$

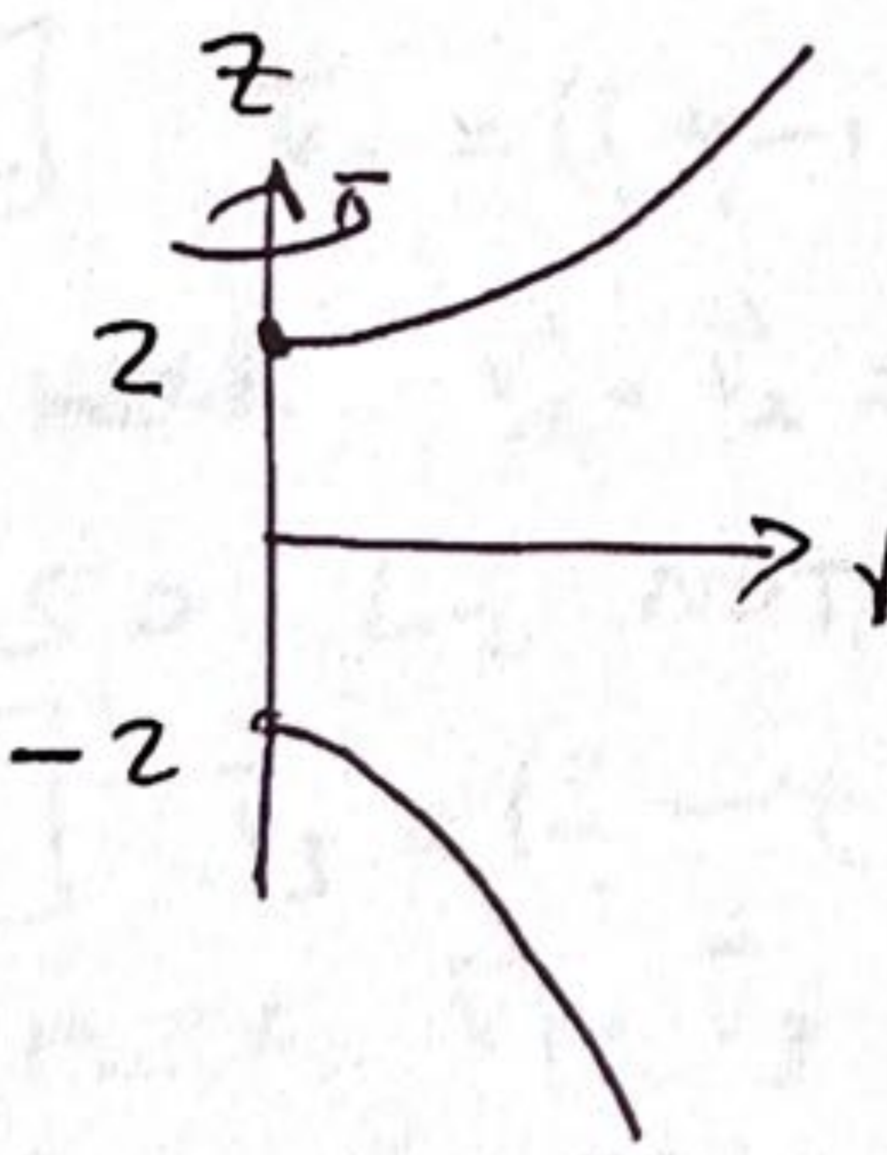
$$y = \rho \sin\phi \sin\theta = 4 \sin \pi/3 \sin(-\pi/4) = -\sqrt{6}$$

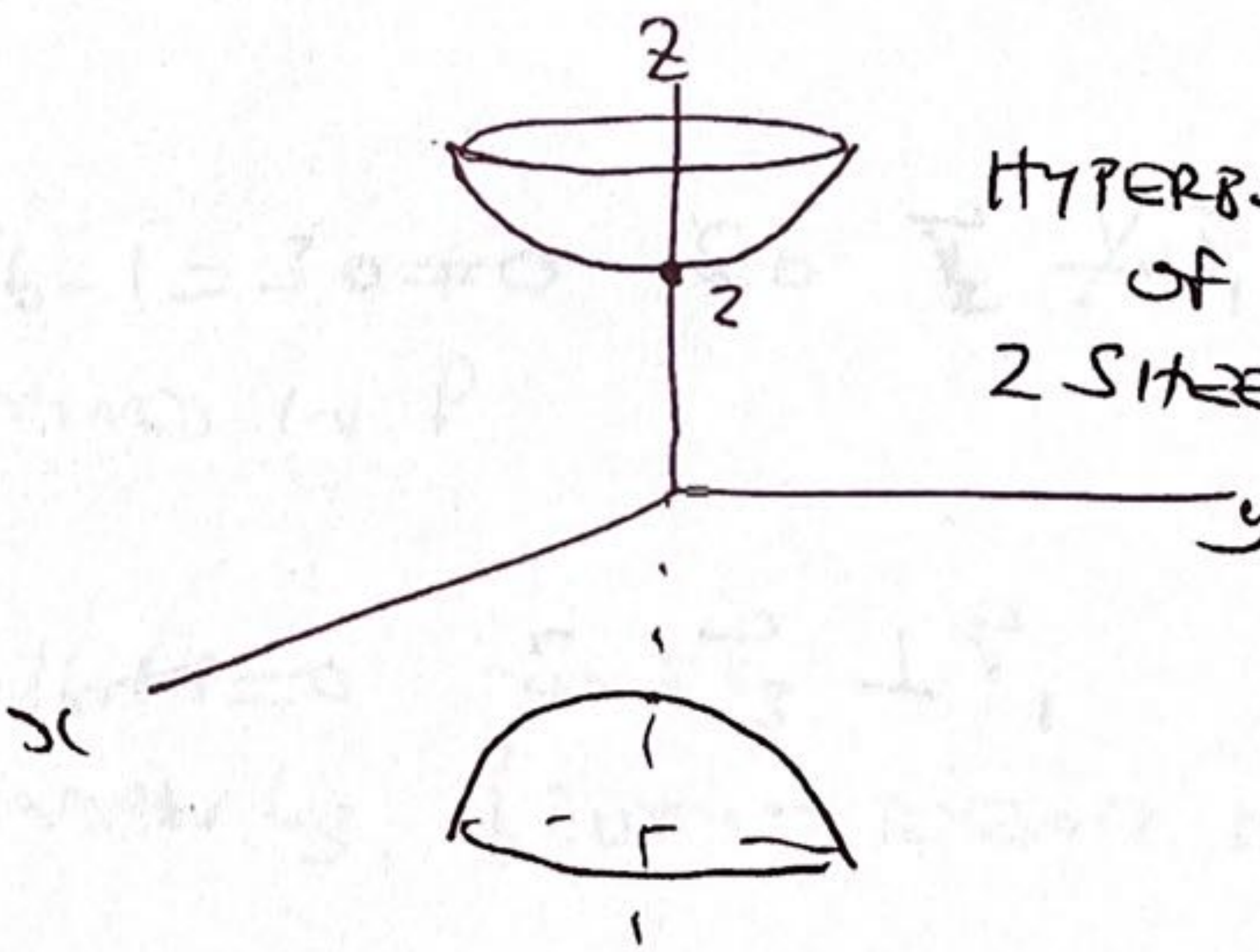
$$z = \rho \cos\phi = 4 \cos \pi/3 = 2$$




$(x, y, z) = (\sqrt{6}, -\sqrt{6}, 2)$

(c) Identify and sketch the surface which is given in cylindrical coordinates by the equation $z^2 - r^2 = 4$.




HYPERBOLOID OF 2 SHEETS

$r = 0 \Rightarrow z = \pm 2$

(5) [12 pts] (a) Let $P$ be the plane parametrized by $\mathbf{r}(s,t) = (1+2s-4t, 3s+t, 6-t)$. Find an equation of the form $Ax+By+Cz=D$ for the plane, $P$.

$\vec{r}(s,t) = \vec{p} + s\vec{v} + t\vec{w}$ WITH $\vec{p} = (1,0,6)$, $\vec{v} = (2,3,0)$

$\vec{w} = (-4,1,-1)$

NORMAL TO PLANE IS $\vec{n} = \vec{v} \times \vec{w}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ -4 & 1 & -1 \end{vmatrix} = (-3, 2, 14)$$

$\vec{r} = (x, y, z)$

<u>EQN</u>

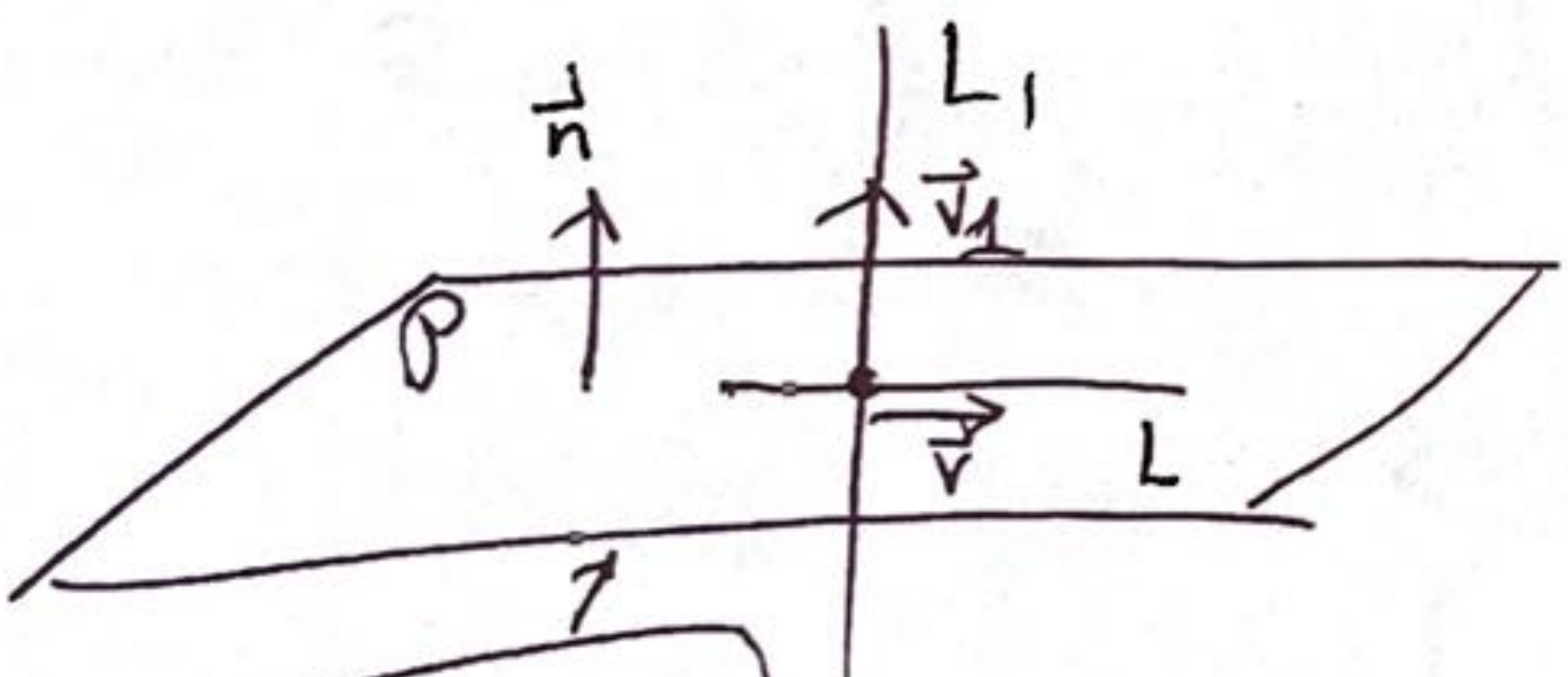
$0 = (\vec{r} - \vec{p}) \cdot \vec{n} = -3(x-1) + 2(y-0) + 14(z-6)$

OR $-3x + 2y + 14z = 81$

(b) Consider the lines, $L_1$, $L_2$, and $L_3$ parametrized by

$L_1 : \mathbf{r}_1(t) = (2+5t, -1+4t, t)$, $= \vec{p}_1 + t\vec{v}_1$   $L_2 : \mathbf{r}_2(t) = (2+3t, 3+4t, 1-t)$,   $L_3 : \mathbf{r}_3(t) = (5+3t, 2-4t, 3+t)$.

Let $\mathcal{P}$ be a plane that is perpendicular to $L_1$. Could $\mathcal{P}$ contain the line $L_2$? Could $\mathcal{P}$ contain the line $L_3$?




$\boxed{\vec{n} \parallel \vec{v}_1}$

IF $L$ IS A ~~PLANE~~ LINE IN $P$

THEN $\boxed{\vec{v} \perp \vec{v}_1}$ MUST HOLD

WHERE $L$ HAS PARM

$\vec{r}(t) = \vec{p} + t\vec{v}$.

NOW $\vec{v}_1 = (5, 4, 1)$

$\boxed{L_2}$ $\vec{v}_2 = (3, 4, -1)$

CHECK $\vec{v}_1 \cdot \vec{v}_2 = 15 + 16 - 1 = 30 \neq 0$ SO $\vec{v}_2 \not\perp \vec{v}_1$

SO $L_2$ NOT CONTAINED IN $P$

$\boxed{L_3}$ $\vec{v}_3 = (3, -4, 1)$

CHECK $\vec{v}_1 \cdot \vec{v}_3 = 15 - 16 + 1 = 0$ SO $\vec{v}_3 \perp \vec{v}_1$

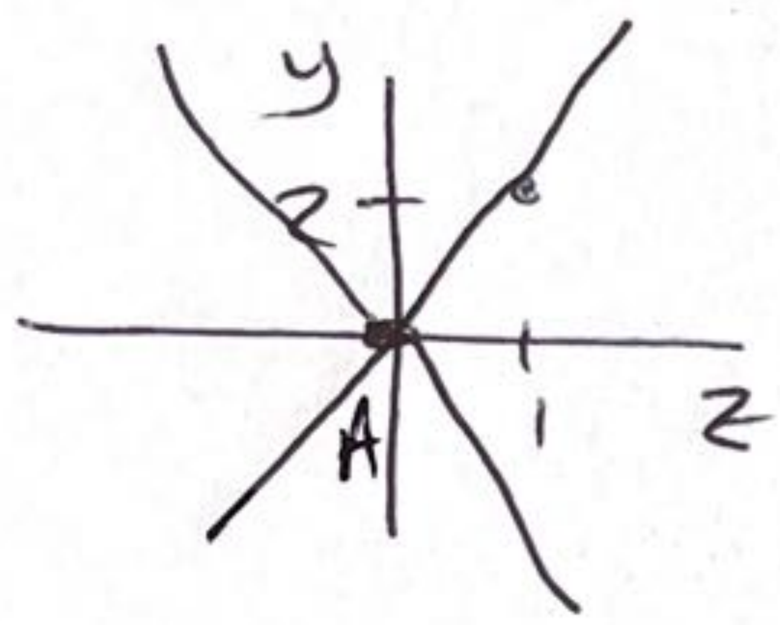
SO $P$ <u>COULD</u> CONTAIN $L_3$. (BUT IT DOESN'T HAVE TO)

(6) [12 pts] Make a labelled sketch of the traces (slices) of the surface

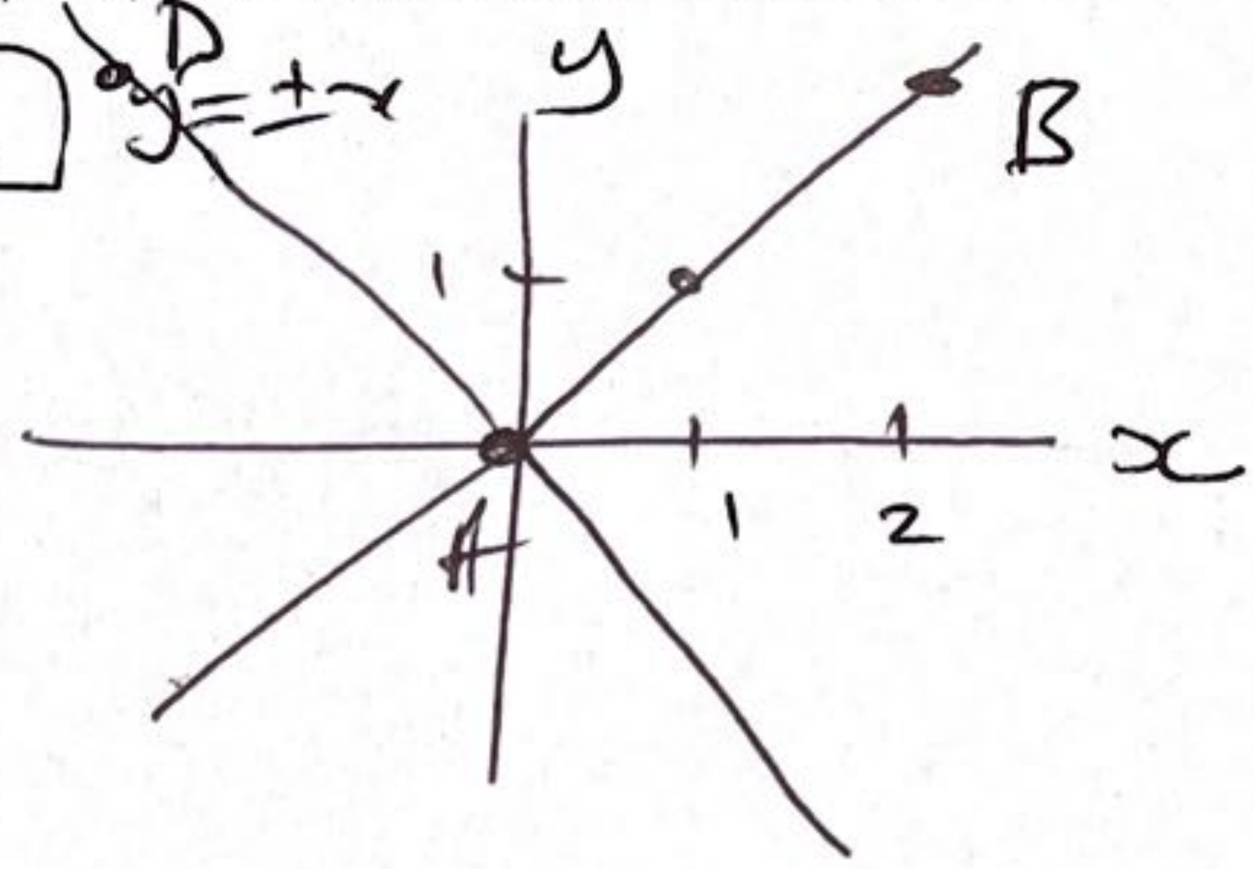
$$x^2 - y^2 + 4z^2 = 0$$

in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Then make a labelled sketch of the surface.

$\boxed{x=0}$ $y = \pm 2z$




$\boxed{z=0}$ $y = \pm x$

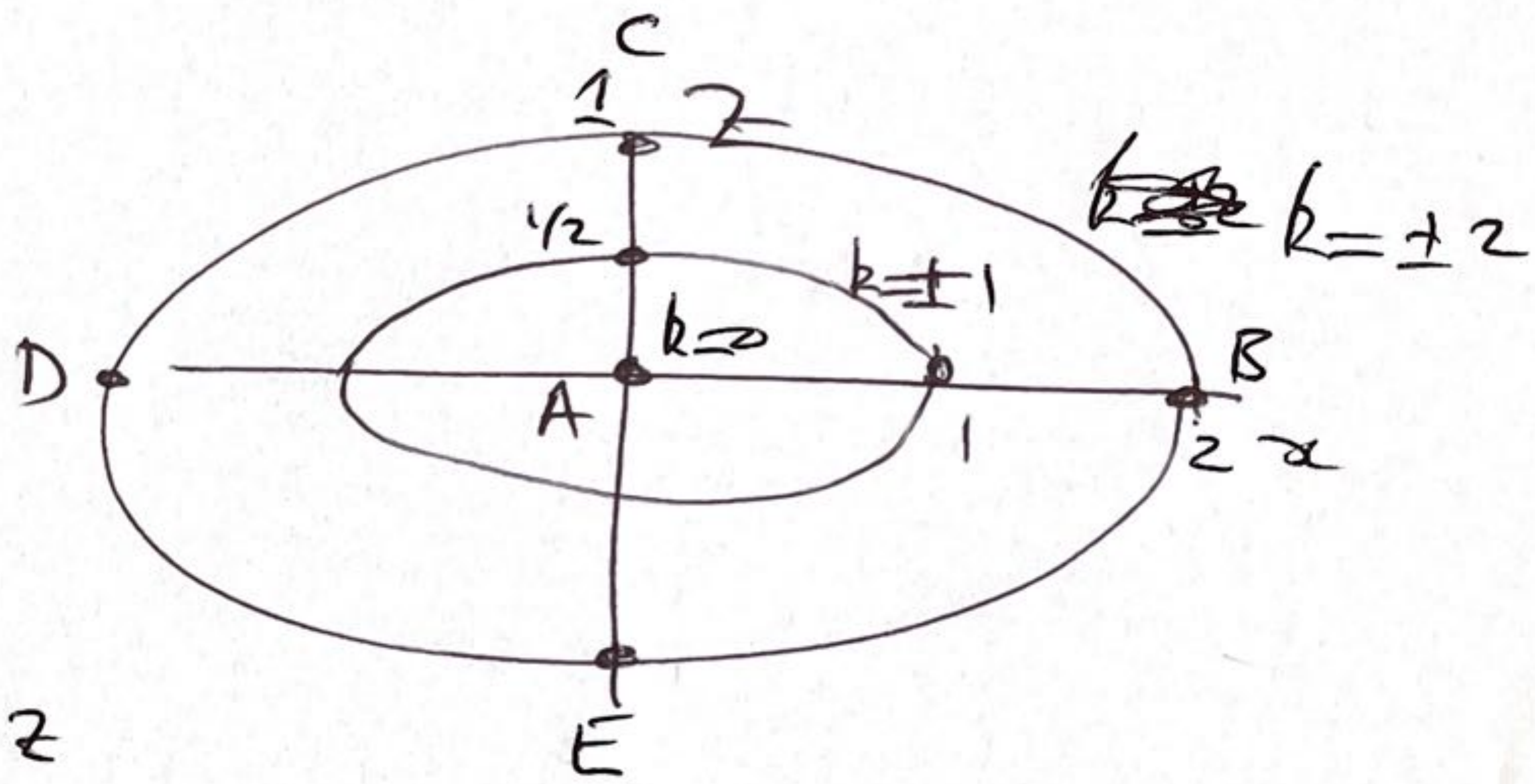


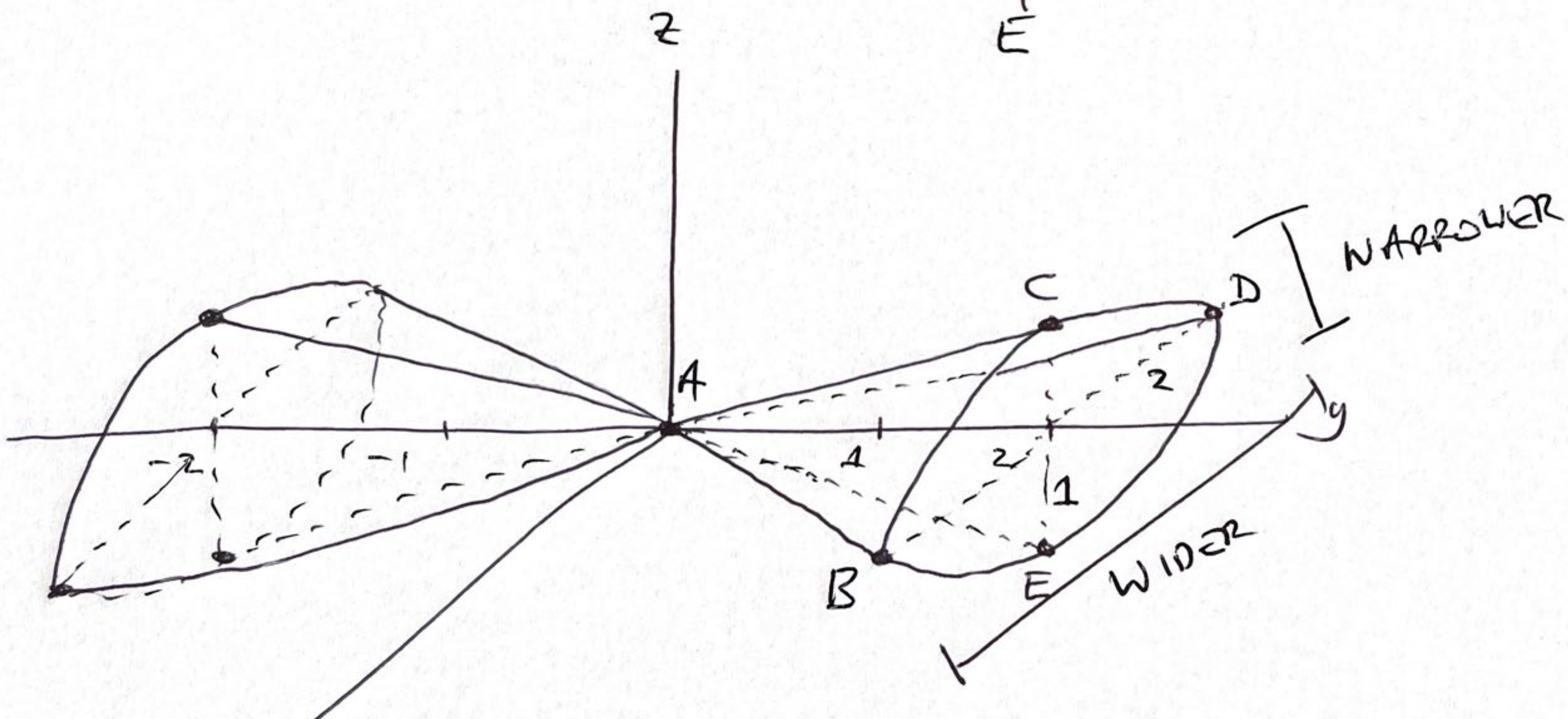

$\boxed{y=k}$ $x^2 + 4z^2 = k^2$

INTERCEPTS

$x=0 \Rightarrow z = \pm \frac{k}{2}$

$z=0 \Rightarrow x = \pm k$




x

DOUBLE ELLIPTICAL CONE

ALIGNED WITH y AXIS.