| LAST NAME: DU CHÂTELET | FIRST NAME: EMILIE | CIRCLE: | Eydelzon | Coskunuzer |
|---|---|---|---|---|
| | | Dahal | Zweck 1pm | Zweck 4pm |

LOOK HER UP ON WIKIPEDIA!

| 1 /10 | 2 /10 | 3 /10 | 4 /10 | 5 /10 | |
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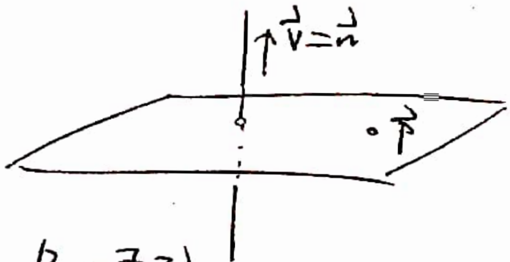
MATH 2415 Final Exam, Fall 2021

No books or notes! **NO CALCULATORS! Show all work and give complete explanations.** This 2 hours 45 mins exam is worth 100 points.

(1) [10 pts]
(a) Find an equation of the form $Ax + By + Cz = D$ for the plane that goes through the point $(5, -1, 0)$ and is perpendicular to the line with parameterization $(x, y, z) = \mathbf{r}(t) = (1 + 2t, 3 - 7t, -2 + 3t)$.

$\vec r(t) = \vec q + t\vec v, \quad \vec v = (2, -7, 3)$




$\vec v = \vec n$ = NORMAL TO PLANE

$\vec p = (5, -1, 0)$ = POINT IN PLANE

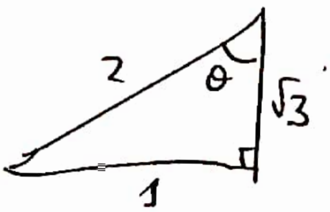
EQN: $0 = (\vec r - \vec p)\cdot \vec n = (x-5, y+1, z)\cdot(2, -7, 3)$

$2(x-5) - 7(y+1) + 3z = 0$

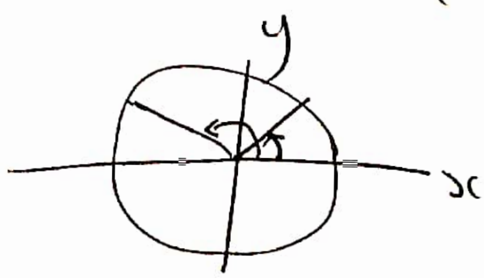
$\boxed{2x - 7y + 3z = 17}$

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors. What are the possible angles between $\mathbf{u}$ and $\mathbf{v}$ if $|\mathbf{u} \times \mathbf{v}| = \frac{1}{2}$?

$\frac{1}{2} = |\vec u \times \vec v| = |\vec u||\vec v| \sin\theta = \sin\theta, \quad 0 \le \theta \le \pi$




$\sin(\pi/6) = \frac{1}{2} = \sin(5\pi/6)$




$\boxed{\theta = \pi/6 \text{ or } 5\pi/6}$

(2) [10 pts] Let $\mathbf{u} = \langle 4, 2, -2\rangle$, $\mathbf{v} = \langle 1, -1, 3\rangle$ and $\mathbf{w} = \langle 1, -1, 1\rangle$.

(a) Find the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.

$$\mathrm{PROJ}_{\vec{w}}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}\frac{\vec{w}}{|\vec{w}|} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$$

$$= \frac{(1,-1,3)\cdot(1,-1,1)}{(1^2+(-1)^2+1^2)}(1,-1,1) = \frac{5}{3}(1,-1,1)$$

(b) Find the area of triangle determined by the vectors $\mathbf{u}$ and $\mathbf{v}$.

$$A = \frac{1}{2}|\vec{u}\times\vec{v}| \qquad \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & -2 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= \frac{1}{2}\sqrt{4^2+(14)^2+6^2} \qquad = 4\vec{i} - 14\vec{j} - 6\vec{k}$$

$$= \frac{1}{2}\sqrt{248}$$

(c) Determine the volume of parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$V = |(\vec{u}\times\vec{v})\cdot\vec{w}|$$

$$= |(4,-14,-6)\cdot(1,-1,1)| \quad \text{from (b)}$$

$$= |4+14-6| = 12$$

(3) [10 pts]

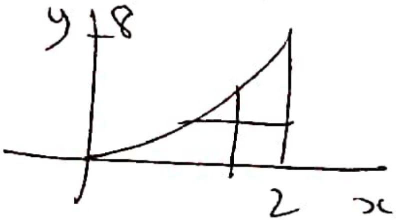
(a) Evaluate the double integral

SWITCH ORDER

$$I=\int_0^8\int_{\sqrt[3]{y}}^2 \cos(x^4)\,dx\,dy.$$

$0 \le y \le 8$

$y^{1/3} \le x \le 2$ Type II




$0 \le x \le 2$

$0 \le y \le x^3$ Type I

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=x^3} \cos(x^4)\,dy\,dx$$

$$= \int_{x=0}^{x=2} \cos(x^4) \int_{y=0}^{3} 1\,dy\,dx$$

$$= \int_0^{2} x^3 \cos(x^4)\,dx \qquad u = x^4, \quad du = 4x^3\,dx$$

$$= \frac{1}{4}\int_0^{16} \cos(u)\,du = \frac{1}{4}\sin(16)$$

(b) Evaluate the line integral $\int_C x e^{yz}\,ds$, where $C$ is the line segment from $(0,0,0)$ to $(1,2,3)$.

$$\vec{r}(t) = t\,(1,2,3), \quad 0 \le t \le 1$$

$$|\vec{r}\,'(t)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\int_C x e^{yz}\,ds = \int_0^1 t\, e^{(2t)(3t)} \sqrt{14}\,dt$$

$$= \sqrt{14}\int_0^1 t e^{6t^2}\,dt \qquad u = 6t^2, \quad du = 12t\,dt$$

$$= \frac{\sqrt{14}}{12}\int_0^6 e^u\,du$$

$$= \frac{\sqrt{14}}{12}\left(e^6 - 1\right)$$

(4) [10 pts] Let $S$ be the surface with parametrization

$$(x, y, z) = \mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, 2\sin\phi\sin\theta, 3\cos\phi), \qquad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

(a) Show that S is an ellipsoid. Hint: Find an equation of the form $F(x, y, z) = 0$ for this surface by eliminating $\phi$ and $\theta$ from the equations for $x$, $y$, and $z$ above.

$$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = \sin^2\phi\cos^2\theta + \sin^2\sin^2\theta + \cos^2\phi$$

$$= \sin^2\phi + \cos^2\phi = 1$$

So $F(x, y, z) = x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 - 1 = 0$

(b) Calculate a normal vector to the ellipsoid at the point where $(\phi, \theta) = (\pi/4, \pi/3)$.

$\vec{n} = \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}$ as $\frac{\partial\vec{r}}{\partial\theta}$ and $\frac{\partial\vec{r}}{\partial\phi}$ are target vectors to the grid curves $\phi = \phi_0$ and $\theta = \theta_0$.

So

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & 2\cos\phi\sin\theta & -3\sin\phi \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sqrt{6}}{4} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{2} & \frac{-3\sqrt{2}}{2} \end{vmatrix} = \left(-\frac{3}{2}, \frac{-3\sqrt{3}}{4}, -1\right)$$

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(5) [10 pts] Use the method of Lagrange multipliers to find the absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2$ on the ellipse $(x+1)^2 + 3y^2 = 4$.

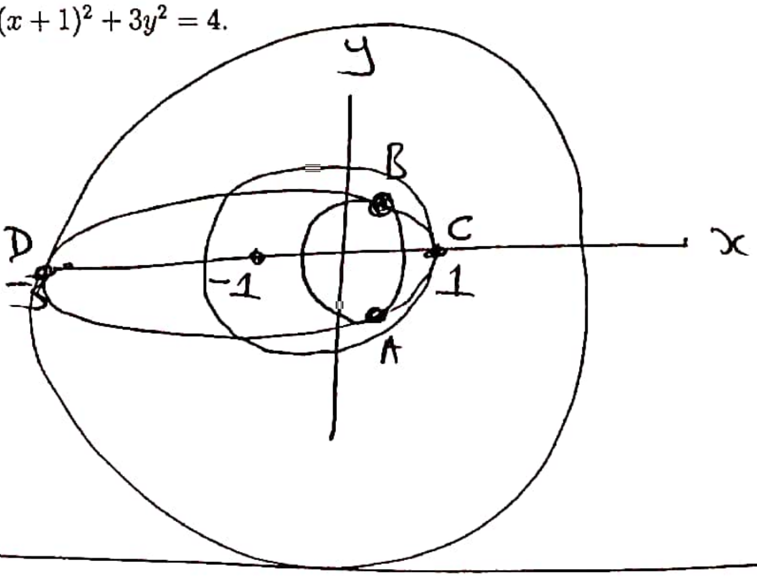
EXPECT 4 CPTS AT

$(1, 0)$

$(-3, 0) \leftarrow$ MAX

$(a, \pm b)$ for some $a > 0$.

$\uparrow$ MIN




$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ : $2x = \lambda\, 2(x+1) \Rightarrow x = \lambda(x+1)$ (1)

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$ : $2y = \lambda\, 6y \Rightarrow y(1-3\lambda) = 0$ (2)

$g = c$ : $(x+1)^2 + 3y^2 = 4$ (3)

By (2) : $y = 0$ OR $\lambda = \frac{1}{3}$

$\boxed{y=0}$ By (3) $(x+1)^2 = 4 \Rightarrow x+1 = \pm 2 \Rightarrow x = -3$ or $1$

$x = 1$ : $\lambda = \frac{x}{x+1} = \frac{1}{2}$ $\quad (1, 0, \frac{1}{2}) = (x\ y\ \lambda)$ $f = 1$

$x = -3$ $\quad \lambda = \frac{-3}{-2} = \frac{3}{2}$ $\quad (-3, 0, \frac{3}{2}) = (x, y, \lambda)$, $f = 9$

$\uparrow$ MAX

$\boxed{\lambda = \frac{1}{3}}$ By (1) $3x = x+1 \Rightarrow x = \frac{1}{2}$

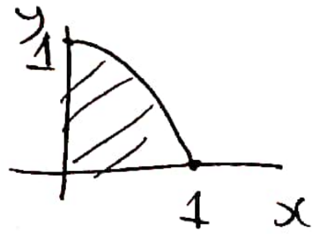
By (3) $\left(\frac{3}{2}\right)^2 + 3y^2 = 4$ $\quad y^2 = \frac{4 - \frac{9}{4}}{3} = \frac{7}{12}$

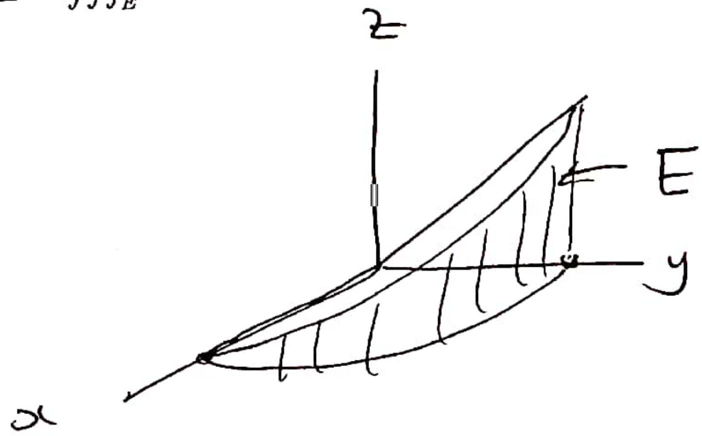
$y = \pm\sqrt{\frac{7}{12}}$ $\quad (x, y, \lambda) = \left(\frac{1}{2}, \pm\sqrt{\frac{7}{12}}, \frac{1}{3}\right)$

$f = \frac{1}{4} + \frac{7}{12} = \frac{10}{12} = 5/6 \leftarrow$ MIN

(6) [10 pts] Let $E$ be the solid in the first octant bounded by the surfaces $z = y$ and $y = 1 - x^2$. (Recall that the first octant is where $x \geq 0$, $y \geq 0$, $z \geq 0$.) Evaluate

$$I = \iiint_E xz \, dV.$$




$0 \leq x \leq 1$

$0 \leq y \leq 1-x^2$

$0 \leq z \leq y$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x^2} \int_{z=0}^{z=y} xz \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} x \int_{y=0}^{1-x^2} \frac{y^2}{2} \, dy \, dx$$

$$= \int_{x=0}^{1} x \left[ \frac{y^3}{6} \right]_{y=0}^{y=1-x^2} dx = \frac{1}{6} \int_0^1 x(1-x^2)^3 \, dx$$

$u = 1-x^2$

$du = -2x\,dx$

$$= \frac{1}{6} \cdot \frac{-1}{2} \int_{u=1}^{u=0} u^3 \, du = \frac{1}{12} \left[ \frac{u^4}{4} \right]_0^1 = \boxed{\frac{1}{48}}$$

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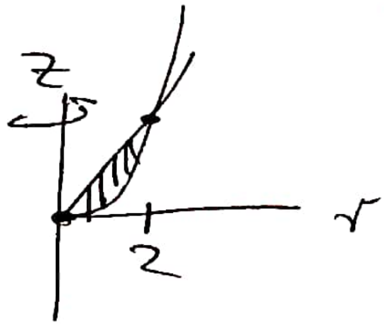
(7) [10 pts] Let $E$ be the solid bounded by the surfaces $z = 2\sqrt{x^2+y^2}$ and $z = x^2 + y^2$. Evaluate the integral

$$I = \iiint_E \sqrt{x^2+y^2}\, dV.$$

CYL COORDS

$z = 2r$

$z = r^2$




Meet at $2r = r^2$

$r(r-2) = 0$, $r = 0, 2$

So $0 \le r \le 2$

$r^2 \le z \le 2r$

$0 \le \theta \le 2\pi$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r^2}^{z=2r} r \cdot r \, dz\, dr\, d\theta$$

$$= 2\pi \int_{r=0}^{2} r^2 (2r - r^2)\, dr$$

$$= 2\pi \left[ \frac{r^4}{2} - \frac{r^5}{5} \right]_0^2$$

$$= 2\pi \left( 8 - \frac{32}{5} \right) = \frac{16\pi}{5}$$

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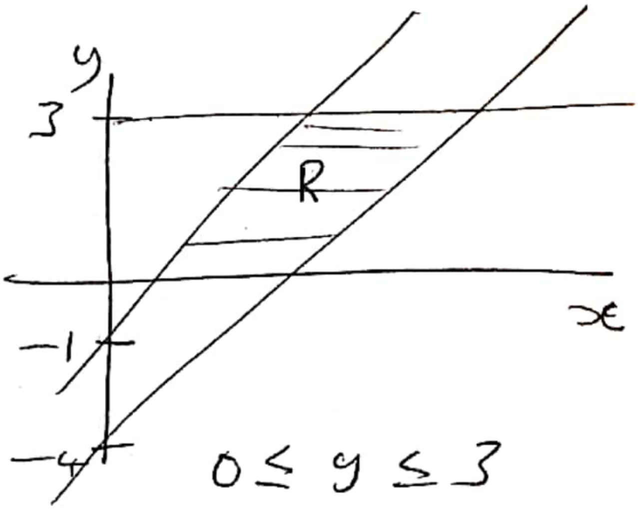
(8) [10 pts] Let $R$ be the domain bounded by the lines $y = 0$, $y = 3$, $y = 2x - 1$, and $y = 2x - 4$. Use the change of variables $u = 2x - y$ and $v = y$ to evaluate

$$\iint_R \frac{\cos y}{(2x-y)^2}\, dA.$$

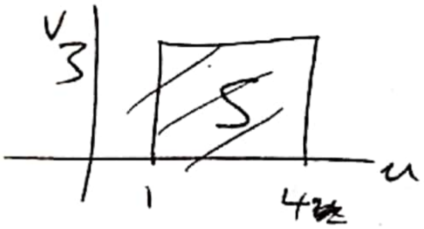



This is a type II region. But integrand is not so nice. So use

$$\boxed{\begin{aligned} u &= 2x - y \\ v &= y \end{aligned}} \quad (*)$$

$0 \le y \le 3$

$\frac{y+1}{2} < x < \frac{y+4}{2}$




$1 \le u \le 4$

$0 \le v \le 3$

as $y = 0 \iff v = 0$

$y = 3 \iff v = 3$

$y = 2x - 1 \iff 2x - y = 1 \iff u = 1$

$y = 2x - 4 \iff 2x - y = 4 \iff u = 4$

SOLVE $(*)$ FOR $x, y$:

$y = v$

$x = \frac{u+v}{2}$

So

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

$$I = \int_{u=1}^{4} \int_{v=0}^{3} \frac{\cos v}{u^2} \cdot \frac{1}{2}\, dv\, du$$

$$= \frac{1}{2} \int_1^4 \frac{1}{u^2}\, du \int_0^3 \cos v\, dv$$

$$= \frac{3}{8} \sin(3)$$

(9) [10 pts] Let $\mathbf{F}(x,y) = \overset{P}{\sin y}\,\mathbf{i} + (\overset{Q}{x\cos y - \sin y})\mathbf{j}$. Verify that line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path. Find a potential function for $\mathbf{F}$ and use it to find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $C$ is any curve from $(2,0)$ to $(1,\pi)$.

(a) $\frac{\partial Q}{\partial x} = \cos y = \frac{\partial P}{\partial y}$. So $\int_C \vec{F}\cdot d\vec{r}$ is path indept

(b) $\nabla f = \vec{F} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\vec{i} + Q\vec{j}$

$\frac{\partial f}{\partial x} = P = \sin y \Rightarrow f = x\sin y + g(y)$

$\frac{\partial f}{\partial y} = Q = x\cos y - \sin y \Rightarrow f = x\sin y + \cos y + h(x)$

So $\boxed{f = x\sin y + \cos y + C}$

(c) $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} \overset{\text{FTCI}}{=} f(1,\pi) - f(2,0)$

$= (1\sin\pi + \cos\pi) - (2\sin 0 + \cos 0)$

$= -1 - 1 = \boxed{-2}$

(10) [10 pts] Let $\mathbf{F}(x,y) = x^3\mathbf{i} - y^3\mathbf{j}$ be the velocity vector field of a fluid flowing in $\mathbb{R}^2$.

(a) Calculate $\nabla \cdot \mathbf{F}$. $= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(-y^3)$

$$= 3x^2 - 3y^2$$

(b) Calculate $\nabla \times \mathbf{F}$. $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & -y^3 & 0 \end{vmatrix}$

$$= 0\vec{i} - 0\vec{j} + \left[\frac{\partial}{\partial x}(-y^3) - \frac{\partial}{\partial y}(x^3)\right]\vec{k} = \vec{0}.$$

(c) On average, is the fluid rotating clockwise, counter-clockwise, or not rotating at all about the point $(1,2)$? Why?

$\nabla \times \vec{F} = \vec{0}$ everywhere + hence at $(1,2)$

So fluid is <u>not</u> rotating about $(1,2)$

(d) On average, is the fluid flowing in, out, or neither in or out, of a small disc centered at $(1,2)$? Why?

$\nabla \cdot \vec{F} = 3\,1^2 - 3\,2^2 = -9 < 0$ @$(1,2)$.

So fluid is flowing <u>in</u> on average.