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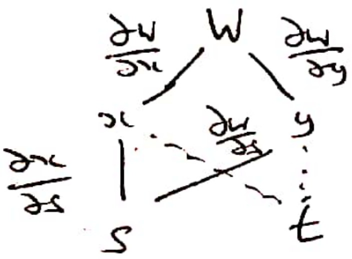
MATH 2415 [Fall 2021] Exam II, Oct 29th

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations**. Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [10 pts]
(a) Suppose that $w = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$. Write the chain rule formula for $\frac{\partial w}{\partial s}$.



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

where $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ are evaluated at $x = g(s,t)$ and $y = h(s,t)$

(b) Let $w = \sin(x^2 + y^2)$, where $x = s^2 t$, $y = st^2$. Use your answer to (a) to find $\frac{\partial w}{\partial s}$ at $(s, t) = (-1, 2)$.

$\frac{\partial w}{\partial x} = 2x \cos(x^2+y^2)$ $\qquad \frac{\partial x}{\partial s} = 2st$

$\frac{\partial w}{\partial y} = 2y \cos(x^2+y^2)$ $\qquad \frac{\partial y}{\partial s} = t^2$

At $(s, t) = (-1, 2)$ we have $x = 2$, $y = -4$

So $\frac{\partial w}{\partial s}(-1,2) = 2.2.\cos(20) . 2.(-1)2 + 2(-4)\cos(20)\, 2^2$

$= -16\cos(20) - 32\cos(20)$

$= -48\cos(20)$

(2) [12 pts] Let $z = f(x,y) = \sqrt{9 + x^2y^2}$

(a) Find an equation of the form $z = Ax + by + C$ for the tangent plane to the surface $z = f(x,y)$ at a point where $x = 2$ and $y = 2$.

$(x_0, y_0) = (2,2)$

$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

$f(2,2) = \sqrt{9+16} = 5$

$\frac{\partial f}{\partial x} = \frac{1}{2}(9+x^2y^2)^{-1/2}\, 2xy^2 = \frac{xy^2}{\sqrt{9+x^2y^2}} = \frac{8}{5}$ @ $(2,2)$

$\frac{\partial f}{\partial y} = \frac{x^2y}{\sqrt{9+x^2y^2}} = \frac{8}{5}$ @ $(2,2)$

$z = 5 + \frac{8}{5}(x-2) + \frac{8}{5}(y-2)$

$\cancel{z = \frac{8}{5}x + \frac{8}{5}y + \frac{-7}{5}}$

(b) Use linear approximation to approximate the value of $f(2.1, 1.8)$.

~~From~~ From (a) for (x,y) near (x_0,y_0)

$f(x,y) \simeq L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

So ~~$f(x,y) \simeq$~~ ~~$f(2,2)+$~~ for (x,y) near $(2,2)$

$f(x,y) \simeq 5 + \frac{8}{5}(x-2) + \frac{8}{5}(y-2)$

$f(2.1, 1.8) \simeq 5 + \frac{8}{5}\,0.1 + \frac{8}{5}(-0.2)$

$= 5 - \frac{8}{5}\frac{1}{10} = \cancel{4.86}\ 4.84$

TRUE ANSWER
$f(2.1, 1.8)$
$= 4.825$

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(3) [12 pts] Let $f(x, y) = (x + 1)y^2e^{-x^2}$.

(a) Calculate the directional derivative of f at the point $(x, y) = (0, 1)$ in the direction of the vector $\mathbf{v} = -\mathbf{i} + \mathbf{j}$.

$u = \frac{1}{\sqrt{2}}(-1, 1)$ is unit vector in direction of $\vec{v}$

$\nabla f = \left(y^2 e^{-x^2} + -2x(x+1)y^2 e^{-x^2},\ 2(x+1)y e^{-x^2}\right)$

$\nabla f = \left((1 - 2x^2 - 2x)y^2 e^{-x^2},\ 2(x+1)y e^{-x^2}\right)$

$\nabla f(0,1) = (1, 2)$

$(D_{\vec{u}} f)(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = (1, 2) \cdot \frac{1}{\sqrt{2}}(-1, 1) = \frac{1}{\sqrt{2}}$

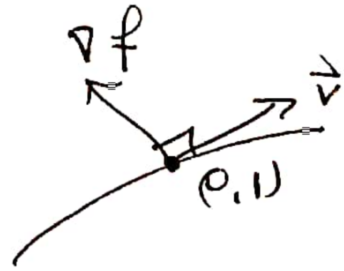
(b) What is the direction of steepest ascent at $(x, y) = (0, 1)$, and what is the rate of change of f in that direction?

$\vec{u} = \frac{\nabla f(0,1)}{|\nabla f(0,1)|} = \frac{(1, 2)}{\sqrt{1^2 + 2^2}} = \frac{1}{\sqrt{5}}(1, 2)$

Rate of Change $= |\nabla f(0,1)| = \sqrt{5}$

(c) Let C be the level curve $f(x, y) = 1$. Find the slope of the tangent line to C at the point $(x, y) = (0, 1)$.

$f(0,1) = 1$, so $(0,1)$ lies on C



$\nabla f(0,1) = (1, 2)$ So $\vec{v} = (2, -1)$

is tangent vector to C @ $(0,1)$ since $\nabla f(0,1) \cdot \vec{v} = 0$

Slope is $m = \frac{\text{RISE}}{\text{RUN}} = \boxed{-\frac{1}{2}}$

(4) [8 pts] Show that the function $f(x,t) = e^{-t}\cos\left(\frac{x}{2}\right)$ satisfies heat equation $f_t = 4f_{xx}$.

$f_t = -e^{-t}\cos\left(\frac{x}{2}\right)$

$f_x = -\frac{1}{2}e^{-t}\sin\left(\frac{x}{2}\right)$

$f_{xx} = -\frac{1}{4}e^{-t}\cos\left(\frac{x}{2}\right)$ 4

So $4f_{xx} = -e^{-t}\cos\left(\frac{x}{2}\right) = f_t$ ✓

(5) [9 pts] Select the answer that is a parametrization of the double cone $x^2 + y^2 = z^2$. *Explain!!*

(I) $(x,y,z) = \mathbf{r}(u,v) = (u, \cos v, \sin v)$ for $-\infty < u < \infty$ and $0 \le v \le 2\pi$

(II) $(x,y,z) = \mathbf{r}(u,v) = (u, v, \sqrt{u^2+v^2})$ for $-\infty < u < \infty$ and $-\infty < v < \infty$

(III) $(x,y,z) = \mathbf{r}(u,v) = (u\cos v, u\sin v, u)$ for $-\infty < u < \infty$ and $0 \le v \le 2\pi$

(I) $x = u,\ y = \cos v,\ z = \sin v$

$x^2+y^2 = u^2 + \cos^2 v \ne z^2 = \sin^2 v$ NO

(II) $x = u \quad y = v,\quad z = \sqrt{u^2+v^2}$

$x^2+y^2 = u^2+v^2, \quad z^2 = u^2+v^2$

So $x^2+y^2 = z^2$. GOOD. BUT $z = \sqrt{u^2+v^2} \ge 0$

which only gives single cone not double cone NO

(III) $x = u\cos v,\ y = u\sin v,\ z = u$

$x^2+y^2 = u^2(\cos^2 v + \sin^2 v) = u^2 = z^2$ GOOD

This time $z = u$ goes from $-\infty < z < \infty$

So double cone

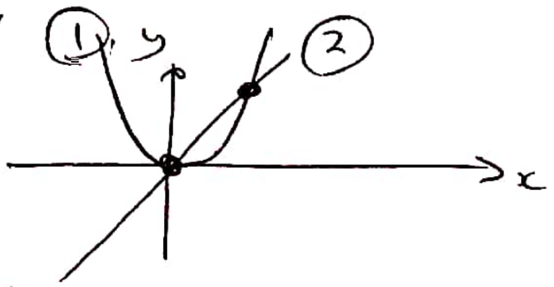
4 3

(6) [12 pts] Find and classify all critical points of the function $f(x,y) = x^3 - 6xy + y^2$.

$$\frac{\partial f}{\partial x} = 3x^2 - 6y = 0 \Rightarrow x^2 = 2y \text{ or } y = \frac{x^2}{2} \quad (1)$$

$$\frac{\partial f}{\partial y} = -6x + 2y \Rightarrow y = 3x. \quad (2)$$

GEOMETRY



2 CRITICAL POINTS

Clearly one is at (0, 0).

Other has $x \geq 0,\ y \geq 0$

ALGEBRA

From (1) and (2) get $3x = \frac{x^2}{2}$; $6x = x^2$

or

$0 = (6-x)x \Rightarrow x = 0$ or $x = 6$

When $x = 0$, $y = 0$ by (2)

When $x = 6$, $y = 18$ by (2)

2 CRITICAL PTS

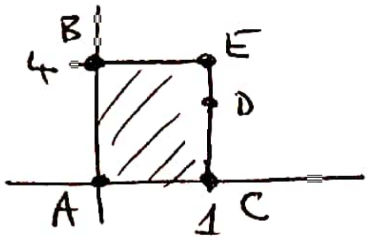
(0, 0)

(6, 18)

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -6 \\ -6 & 2 \end{vmatrix} = 12x - 36 = 12(x-3)$$

C PT	D	f_{xx}	CLASSIFICATION
(0, 0)	$-36 < 0$	*	SADDLE POINT
(6, 18)	$36 > 0$	$36 > 0$	LOCAL MIN.

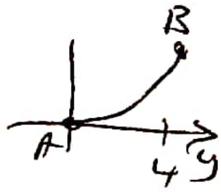
(7) [12 pts] Find the absolute maximum and absolute minimum of the function $f(x,y) = x^3 - 6xy + y^2$ on the rectangle $0 \le x \le 1$, $0 \le y \le 4$. [*You may use your answer to Question (6).*]



- CRITICAL POINTS INSIDE DOMAIN: NONE. FROM ~~(#6)~~.
- FIND CRITICAL POINTS ON BOUNDARY

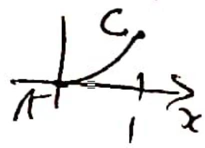
① $x = 0$, $0 \le y \le 4$ $\quad g(y) = f(0,y) = y^2$

$g'(y) = 2y = 0$ @ $y = 0$ only



② $y = 0$, $0 \le x \le 1$ $\quad h(x) = f(x,0) = x^3$

$h'(x) = 3x^2 = 0$ @ $x = 0$ only

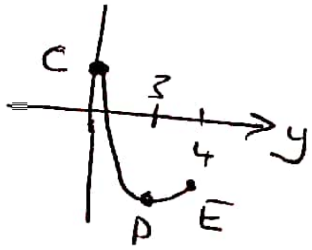


③ $x = 1$, $0 \le y \le 4$, $\quad k(y) = f(1,y) = 1 - 6y + y^2$

$k'(y) = -6 + 2y = 0$ @ $y = 3$.

$k(0) = 1$, $\quad k(4) = 1 - 24 + 16 = -7$

$k(3) = 1 - 18 + 9 = -8$



④ $y = 4$, $0 \le x \le 1$

$l(x) = f(x,4) = x^3 - 24x + 16$

$l'(x) = 3x^2 - 24 = 3(x^2 - 8) = 0$

@ $x = \pm 2\sqrt{2} \simeq \pm 2.82$

These points are not in interval $[0,1]$

ENDPOINTS are B, E calculated before

LABEL	(x,y)	f
A	$(0,0)$	0
B	$(0,4)$	16 MAX
C	$(1,0)$	1
D	$(1,3)$	-8 MIN
E	$(1,4)$	-7