SOLUTIONS LAST NAME: GIBBS	FIRST NAME: WILLARD	CIRCLE:	Eydelzon	Coskunuzer
		Dahal	Zweck 1pm	Zweck 4pm

(LOOK HIM UP ON WIKIPEDIA)

1 /12	2 /12	3 /13	4 /12	5 /13	6 /13	T /75

MATH 2415 [Fall 2021] Exam I, Oct 1st

No books or notes! **NO CALCULATORS!** Show all work and give **complete explanations.** Don't spend too much time on any one problem. This 75 minute exam is worth 75 points.

(1) [12 pts] Let $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$

(a) Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\text{PROJ}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{(-1,2,-3)\cdot(5,2,0)}{5^2+2^2+0^2}(5,2,0)$$

$$= \frac{-1}{29}(5,2,0)$$

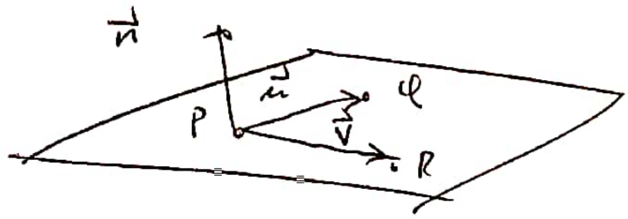
(b) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Is $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} \le \theta < \pi$? Why?

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$\vec{u}\cdot\vec{v} = -1 < 0$

So $\cos\theta < 0$

So $\boxed{\pi/2 \le \theta < \pi}$



(2) [12 pts] Let $\mathcal{P}$ be the plane that contains the points $P = (1, 0, 2)$, $Q = (4, 1, -2)$, and $R = (2, 0, 0)$.
(a) Find two different vectors that are perpendicular to the plane $\mathcal{P}$ and have length 6.

$$\vec{u} = \overrightarrow{PQ} = Q - P = (4, +1, -2) - (1, 0, 2) = (3, 1, -4)$$

$$\vec{v} = \overrightarrow{PR} = R - P = (2, 0, 0) - (1, 0, 2) = (1, 0, -2)$$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -4 \\ 1 & 0 & -2 \end{vmatrix} = (-2, 2, -1)$$

$$\|\vec{n}\| = \sqrt{4+4+1} = 3$$

So to get vectors of length 6 use

$$\vec{n}_{\pm} = \pm 2\vec{n} = \pm(-4, 4, -2)$$

$$\boxed{\vec{n}_{\pm} = \pm(-4, 4, -2)}$$

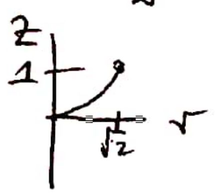
(b) Find the area of the triangle whose vertices are P, Q, and R.

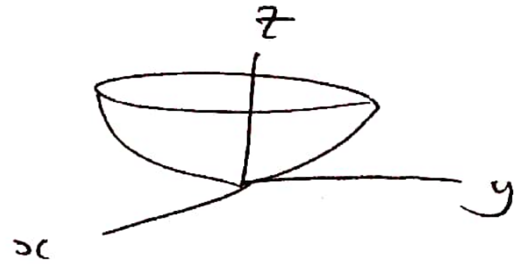
$$A = \frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{3}{2}$$ from above

(3) [13 pts]

(a) Sketch the surface given in cylindrical coordinates by $r = \sqrt{2z}$.

$z = \frac{r^2}{2}$





(b) Convert the point $(-\sqrt{3}, \sqrt{3}, \sqrt{2})$ from rectangular coordinates to spherical coordinates.

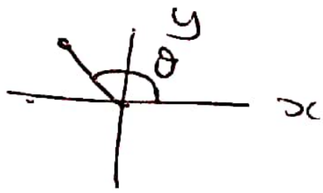
$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

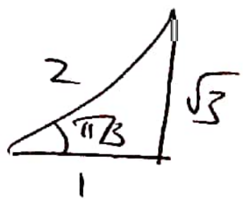
$\rho = \sqrt{3+3+2} = \sqrt{8}$ $\boxed{\rho = 2\sqrt{2}}$

$\boxed{\theta = \frac{3\pi}{4}}$



$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{2}}{\sqrt{8}} = \frac{1}{2}$

$\boxed{\phi = \pi/3}$

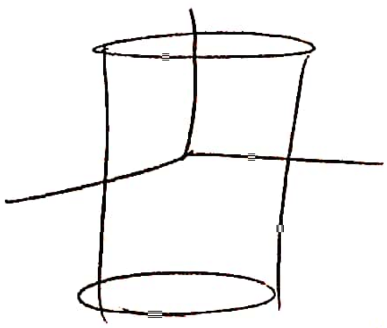


(c) Find an equation in rectangular coordinates for the surface given in spherical coordinates by $\rho \sin\phi = 1$. Describe the surface.

$x^2 + y^2 = \rho^2 \sin^2\phi (\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\phi = 1$

$\boxed{x^2 + y^2 = 1}$

cylinder



(4) [12 pts]

(a) Let C be the curve parameterized by

$$\mathbf{r}(t) = \ln(t+1)\mathbf{i} + e^t\mathbf{j} + 2\cos t\mathbf{k}.$$

Find the parametric equation of the tangent line to C at the point $P = (0, 1, 2)$.

$\vec{P} = (0, 1, 2) \qquad \vec{r}(0) = (0, 1, 2) = \vec{P}$

$\vec{r}'(t) = \frac{1}{t+1}\vec{i} + e^t\vec{j} - 2\sin t\,\vec{k}$

$\vec{v} = \vec{r}'(0) = \vec{i} + \vec{j} \qquad \vec{v} = (1, \cancel{0}, 0)$

$\vec{\ell}(s) = \vec{P} + s\vec{v} = (0, 1, 2) + s(1, 1, 0)$

$= (s, 1+s, 2)$

(b) Find the length of the curve $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$ between the points $P = (2, 0, 0)$ and $Q = (-2, 0, \pi)$.

$\vec{r}(0) = P$

$\vec{r}(\pi) = Q$

$\vec{r}'(t) = (\cancel{2}\sin t, 2\cos t, 1)$

$|\vec{r}'(t)| = \sqrt{4+1} = \sqrt{5}$

$L = \int_0^\pi \sqrt{5}\,dt = \sqrt{5}\pi$

(5) [13 pts]

(a) Find an equation for the plane that goes through the point $(1, -3, 2)$ and is parallel to the plane whose equation is $x + 4y + 5z = 0$.

Parallel planes have the same normals

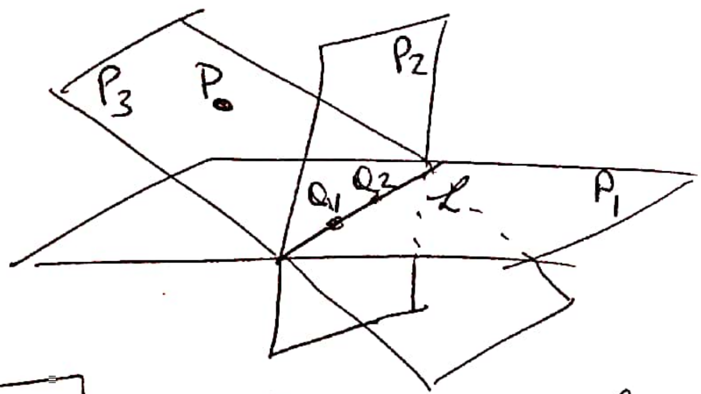
So our plane goes through point $\vec{p} = (1, -3, 2)$ and has normal $\vec{n} = (1, 4, 5)$. It's eq^n is

$0 = (\vec{r} - \vec{p}) \cdot \vec{n} = (x-1, y+3, z-2) \cdot (1, 4, 5)$

$0 = x - 1 + 4(y+3) + 5(z-2)$

$\boxed{x + 4y + 5z = -1}$

(b) Let $\mathcal{L}$ be the line of intersection of the planes $x+y+z = 3$ and $x-y+4z = 5$. Find a parameterization of the plane that contains the point $P = (1, 2, 0)$ and the line $\mathcal{L}$.



P_1 : $x+y+z=3$

P_2 : $x-y+4z=5$

FIND ~~equation~~ parametrization of P_3

$\boxed{\text{STEP 1}}$ FIND 2 POINTS ON $\mathcal{L}$.

$\boxed{Q_1}$ Set $z=0$ in eq^{ns} of planes to get $\boxed{\begin{matrix} x+y=3 \\ x-y=5 \end{matrix}}$

Solve to get $Q_1 = (4, -1, 0)$

$\boxed{Q_2}$ set $z=1$ to get $\begin{matrix} x+y = 2 \\ x-y = 1 \end{matrix}$ So $Q_2 = \left(\frac{3}{2}, \frac{1}{2}, 1\right)$

$\boxed{\text{STEP 2}}$ FIND 2 VECTORS IN P_3 : $\vec{u} = \overrightarrow{PQ_1} = (4, -1, 0) - (1, 2, 0)$
$= (3, -3, 0)$

$\vec{v} = \overrightarrow{PQ_2} = \left(\frac{3}{2}, \frac{1}{2}, 1\right) - (1, 2, 0)$
$= (1/2, -3/2, 1)$

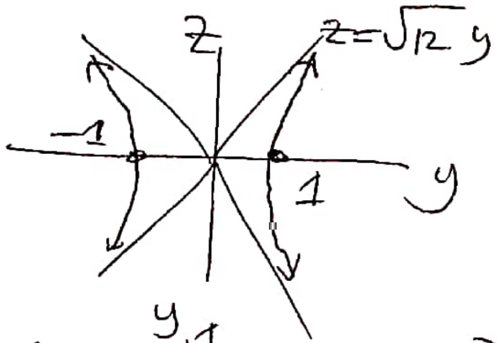
$\boxed{\text{STEP 3}}$ $\boxed{\vec{r}(s, t) = (1, 2, 0) + s(3, -3, 0) + t(1/2, -3/2, 1)}$

(6) [13 pts] Make a labelled sketch of the traces (slices) of the surface

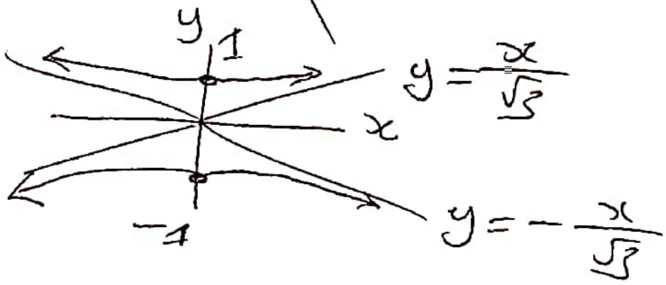
$$y^2 - \frac{x^2}{3} - \frac{z^2}{12} = 1$$

in the planes $x = 0$, $z = 0$, and $y = k$ for $k = 0, \pm 1, \pm 2$. Then make a labelled sketch of the surface.

$\boxed{x=0}$ $\quad y^2 - \left(\frac{z}{\sqrt{12}}\right)^2 = 1$



$\boxed{z=0}$ $\quad y^2 - \left(\frac{x}{\sqrt{3}}\right)^2 = 1$



$\boxed{y=0}$ EMPTY SET

$\boxed{y=\pm 1}$ ORIGIN

$\boxed{y=\pm 2}$ $\quad \frac{x^2}{3} + \frac{z^2}{12} = 3 \Rightarrow \frac{x^2}{9} + \frac{z^2}{36} = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{z}{6}\right)^2 = 1$