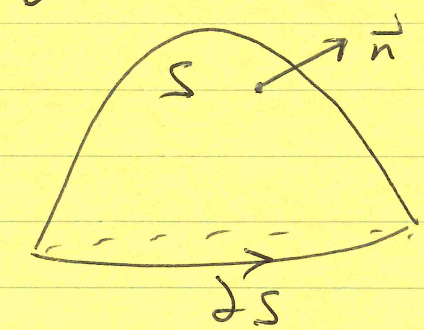
16.8 STOKES' THM

Let S be an oriented surface with positively oriented ~~unit~~ unit normal VF $\vec{n}$.

Suppose S has a boundary curve, ∂S.

EG S = Northern Hemisphere outward normal.

∂S = Equator



We choose the orientation for ∂S (direction to go around curve ∂S) as follows:

~~Walk around ∂S with your head pointing in dirn of $\vec{n}$. in the direction which~~

The direction you should walk around ∂S is the one such that if your head points in dirn of $\vec{n}$, the the surface S will be on your left.

STOKES' THM Let S, ∂S be as above. Let $\vec{F}$ be VF on $\mathbb{R}^3$. Then

$$\boxed{\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}}$$

NOTES

1. If $S = D$ is region in plane $\mathbb{R}^2$ ~~then~~ and $\vec{F}$ is a VF on $\mathbb{R}^2$ $(\vec{F} = P\vec{i} + Q\vec{j})$ Then Stokes' Thm becomes Green's Thm!!

2. $$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int_{\partial S} (\vec{F} \cdot \vec{T})\, ds$$

Surface Integral of normal cpt of $\nabla \times \vec{F}$ over S

= Line Integral around ∂S of tangential cpt of $\vec{F}$.

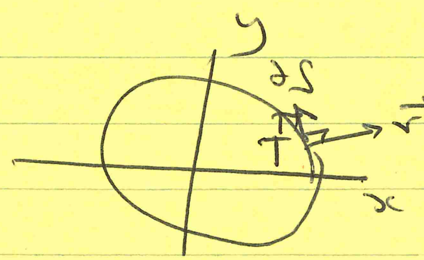
EXS

1. Let S be sphere, $\vec{F}$ any VF.
Then
$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0 \quad \text{as } \partial S = \emptyset.$$

2. Let S be northern hemisphere and let
$\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + \cancel{z\vec{k}}\; z\cos^2(x)e^{y}\vec{k}$.

Then
$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} (\vec{F} \cdot \vec{T})\, ds$$

~~On ∂S we have $z = 0$.~~
~~$\vec{T} = f(x,y)\vec{i} + g(x,y)\vec{j}$~~
~~So $\vec{F} \cdot \vec{T}$~~



On ∂S, $z = 0$ so

$$\vec{F}(x,y,0) = x\vec{i} + y\vec{j} + 0\vec{k} = \vec{r}$$

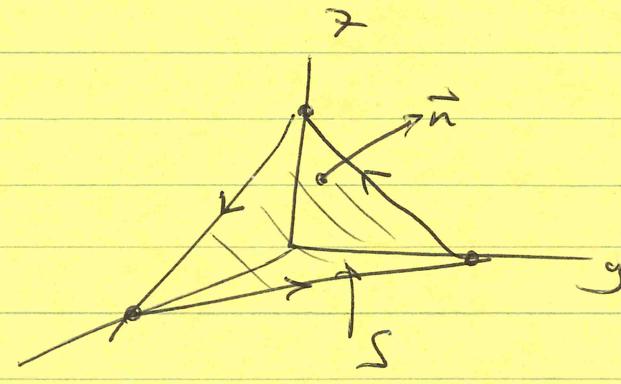
and $\vec{r} \cdot \vec{T} = 0$.

So. $\displaystyle\int_{\partial S} (\vec{F}\cdot\vec{T})\, ds = 0.$

3 $\vec{F}(x,y,z) = (x+y^2)\vec{i} + (y+z^2)\vec{j} + (z+x^2)\vec{k}$.

C = Triangle Vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ oriented in that order.

Find $\displaystyle\int_C \vec{F}\cdot d\vec{r}$.



$C = \partial S$

where S is Shaded Δ.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$$

$$= -2z\vec{i} - 2x\vec{j} - 2y\vec{k}$$

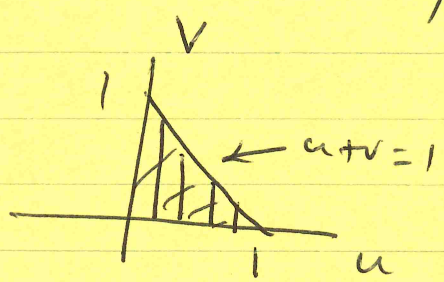
$$= -2(z\vec{i} + x\vec{j} + y\vec{k})$$

<u>Parametrize</u> S as follows.

S is part of plane $x + y + z = 1$. (3 vertices lie in this plane)

Use

$$x = u$$
$$y = v$$
$$z = 1 - u - v$$



$$0 \le u \le 1$$
$$0 \le v \le 1 - u$$

$$\vec{r}(u,v) = (u, v, 1-u-v)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} \checkmark$$

(Agrees $\bar{c}$ $\vec{n}$ from plane)
OUTWARD/UPWARD //

So

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= -2 \int_{u=0}^{1} \int_{v=0}^{1-u} (1-u-v, u, v) \cdot (1,1,1) \, dv \, du$$

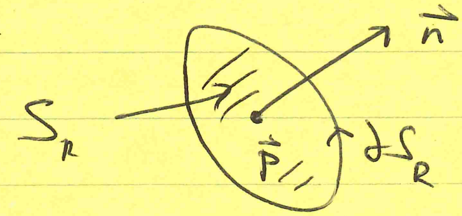
$$= -2 \int_{u=0}^{1} \int_{v=0}^{1-u} 1 \, dv \, du = -2 \, \text{Area}(\triangle) = -2 \cdot \tfrac{1}{2} \cdot 1 \cdot 1$$

$$= -1 \checkmark$$

MORE ON PHYSICAL MEANING OF CURL

Let $\vec{F}$ be velocity VF of fluid in $\mathbb{R}^3$.

Let $\vec{p} \in \mathbb{R}^3$ and let $\vec{n}$ be a vector at $\vec{p}$.



Let S_R be disc $\bar{c}$ center $\vec{p}$, normal $\vec{n}$, radius R.

Orient the circle ∂S_R as in Stokes' Thm.

The Angular Velocity around ∂S_R is defined to be

$$\frac{\text{Component of } \vec{F} \text{ tangent to } \partial S_R}{R} \qquad \left[\omega = \frac{v}{R}\right]$$

So Average Angular Velocity around ∂S_R

$$= \frac{1}{\text{LENGTH}(\partial S_R)} \int_{\partial S_R} \frac{\vec{F} \cdot \vec{T}}{R} \, ds$$

$$= \frac{1}{2\pi R^2} \int_{\partial S_R} \vec{F} \cdot d\vec{r} = \frac{1}{2} \, \frac{\int_{\partial S_R} \vec{F} \cdot d\vec{r}}{\text{AREA}(S_R)}$$

<u>UNITS</u> RAD/SEC.

~~THM~~

$$= \frac{1}{2} \frac{\iint_{S_R} (\nabla \times \vec{F}) \cdot \vec{n} \, dS}{\text{AREA}(S_R)} \quad \text{by STOKES' THM}$$

$$\underset{R \text{ SMALL}}{\simeq} \frac{1}{2} \frac{\left(\nabla \times \vec{F}(\vec{p}) \cdot \vec{n} \right) \text{AREA}(S_R)}{\text{AREA}(S_R)}$$

Assuming $\nabla \times \vec{F}$ is approx const on S_R

$$= \frac{1}{2} \left(\nabla \times F(\vec{p}) \right) \cdot \vec{n}.$$

TWO CASES

① If $(\nabla \times \vec{F})(\vec{p}) = \vec{0}$ Then fluid doesn't rotate about $\vec{p}$, no matter what axis $\vec{n}$ is chosen to measure angular velocity.

Say $\vec{F}$ is IRROTATIONAL

② If $(\nabla \times \vec{F})(\vec{p}) \neq \vec{0}$ Then the axis $\vec{n}$ for which Average Angular Velocity about that axis is LARGEST is given by

$$\vec{n} = \frac{(\nabla \times \vec{F})(\vec{p})}{|(\nabla \times \vec{F})(\vec{p})|}$$

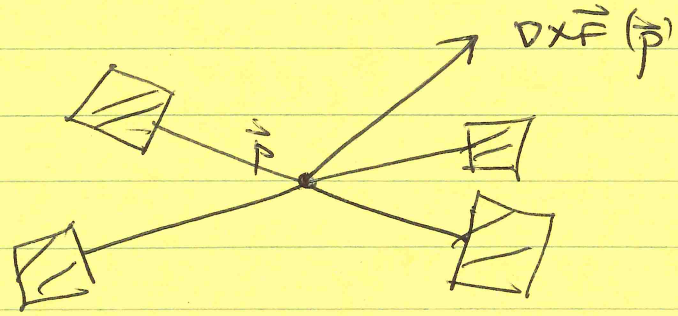
and Av. Angular Velocity about that axis is

$$\frac{1}{2}\left|(\nabla\times\vec{F})(\vec{p})\right|.$$

<u>PHYSICALLY</u> Put Paddlewheel in fluid.

Axis will align with $(\nabla\times\vec{F})(\vec{p})$.

Angular Velocity of PW is $\frac{1}{2}\left|(\nabla\times\vec{F})(\vec{p})\right|$

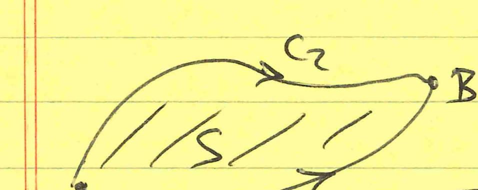


APPLICATION TO CONSERVATIVE VFs

Suppose $\nabla\times\vec{F}=\vec{0}$. Then $\vec{F}=\nabla f$ is conservative

<u>Reason</u> $\int_C \vec{F}\cdot d\vec{r}$ is indept of path

<u>IDEA</u>



Let C_1, C_2 be 2 curves from A, B and S surface with $\partial S = C_1 - C_2$ (as in picture).

Then

$$0=\iint_S(\nabla\times\vec{F})\cdot d\vec{S} = \int_{C_1}\vec{F}\cdot d\vec{r} - \int_{C_2}\vec{F}\cdot d\vec{r}$$