15.3 DOUBLE INTEGRALS IN POLAR COORDS

IDEA When calculate $\iint_D f(x,y)\, dA$

If D and/or f is simpler in polar coords use

PC $\iint_D f(x,y)\, dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

$$\boxed{dA = r\, dr\, d\theta}$$

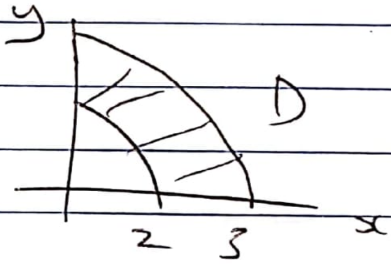
BE A PIRATE!

DON'T FORGET:

r = AREA STRETCHING FACTOR

EXS

1 $I = \iint_D y\, dA$



$0 \le \theta \le \pi/2$

$2 \le r \le 3$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=2}^{3} r\sin\theta\, r\, dr\, d\theta$$

$$= \left[\int_0^{\pi/2} \sin\theta\, d\theta\right] \left[\int_2^3 r^2\, dr\right]$$

TRICK!

$$= \left[-\cos\theta\right]_0^{\pi/2} \left[r^3/3\right]_2^3 = \frac{19}{3}$$

$$\int_{u=a}^{u=b}\int_{v=c}^{v=d} g(u)\, h(v)\, dv\, du$$

$$= \int_{u=a}^{u=b} g(u) \left(\int_{v=c}^{v=d} h(v)\, dv\right) du$$

$$= \left[\int_{u=a}^{u=b} g(u)\, du\right]\left[\int_{v=c}^{v=d} h(v)\, dv\right]$$

Need constant limits of integration and $f(u,v) = g(u)\, h(v)$

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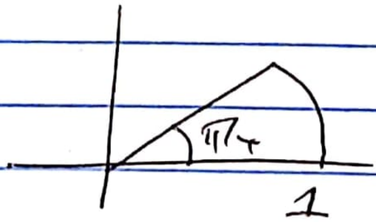
(3) $\iint_D (x^2+y^2)\, dA$



$0 \le \theta \le \pi/4$

$0 \le r \le 1$

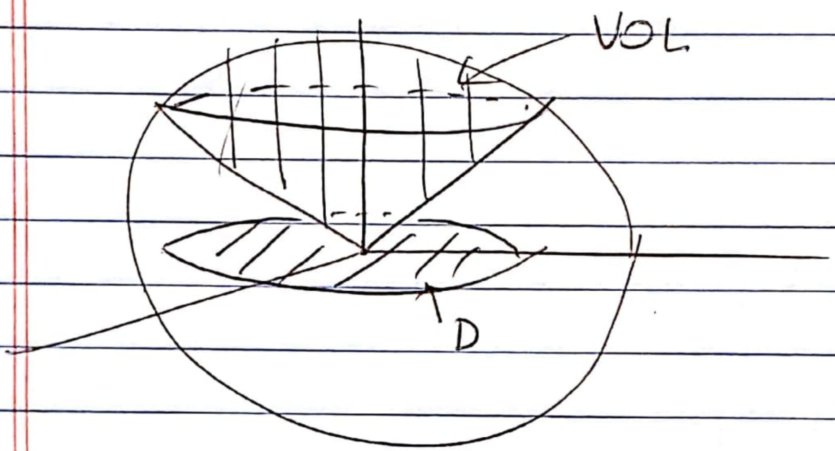
$$= \int_0^{\pi/4}\int_0^1 r^2\, r\, dr\, d\theta$$

$$= \left(\int_0^{\pi/4} d\theta\right)\int_0^1 r^3\, dr$$

$$= \frac{\pi}{4}\left[\frac{r^4}{4}\right]_0^1 = \frac{\pi}{16}$$

3 Find VOL above cone $z = \sqrt{x^2+y^2}$
and below sphere $x^2+y^2+z^2 = 1$.

VOL of Icecream + Cone

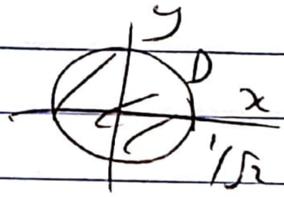


$$VOL = \iint_D f(x,y)\, dA \qquad f = \text{HEIGHT FROM CONE TO SPHERE}$$

CONE $\quad z = r$

SPHERE $\quad r^2+z^2 = 1, \qquad z = \sqrt{1-r^2}$

INTERSECTION $\quad 2r^2 = 1, \qquad r = 1/\sqrt{2}$



D IS $\quad 0 \le \theta \le 2\pi$

$\qquad\quad 0 \le r \le 1/\sqrt{2}$

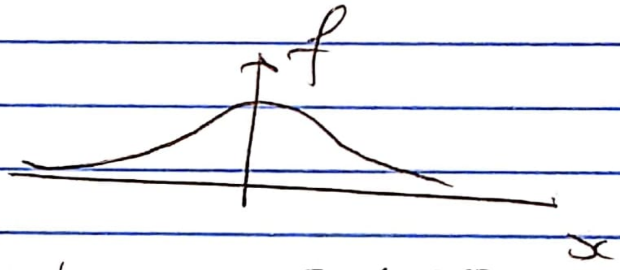
$$f(r,\theta) = \sqrt{1-r^2} - r$$

$$\text{VOL} = \int_0 \int_0 \left(\sqrt{1-r^2} - r\right) r \, dr \, d\theta$$

$$\text{VOL} = \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

④ THM
$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}}$$

$$f(x) = e^{-x^2}$$



- NORMAL DISTRIBUTION in STATISTICS
- An example of a Probability Distribution which is a continuous version of a histogram.

PROPERTIES OF PROBABILITY DISTRIBUTIONS

① $f(x) \geq 0$. Probs are non neg.

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$ Total Prob is 1.

Then $\text{Prob}(a < x < b) = \int_a^b f(x)\,dx$

PF OF THM

TRICK Do $I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA$ two ways

RECT COORDS

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right)$$

So $\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{I}}$

POLAR COORDS

$$I = \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

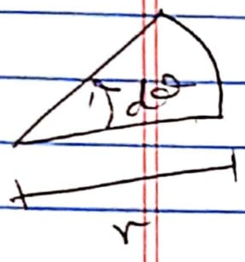
$$= 2\pi \int_0^{\infty} r e^{-r^2} dr \qquad u = r^2, \quad du = 2r\,dr$$

$$= \pi \int_0^{\infty} e^{-u}\, du = \pi$$

So $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{I} = \sqrt{\pi}$ $\square$

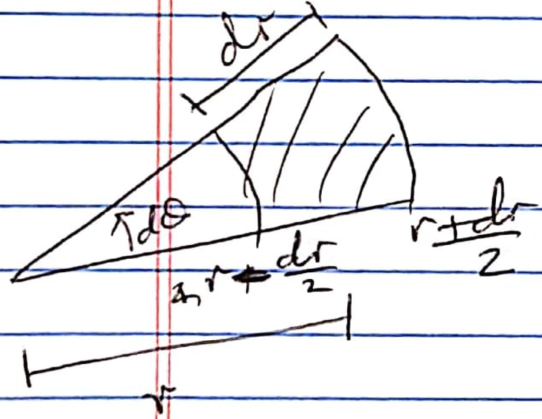
WHY IS $dA = r\,dr\,d\theta$?

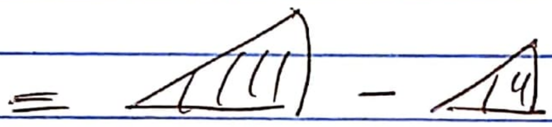
AREA OF WEDGE



$$A = \underbrace{\frac{d\theta}{2\pi}}_{\text{FRACTION OF CIRCLE}} \underbrace{\pi r^2}_{\text{AREA OF CIRCLE}} = \frac{1}{2} r^2 d\theta$$

AREA OF POLAR RECTANGE





$$dA = \frac{1}{2}\left(r + \frac{dr}{2}\right)^2 d\theta - \frac{1}{2}\left(r - \frac{dr}{2}\right)^2 d\theta$$

$$= r\,dr\,d\theta$$