12.4 THE CROSS PRODUCT

Recall 2x2 DETERMINANT $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

DEF (CS) THE **CROSS PRODUCT** of 2 vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$ 3x3 DETERMINANT

$$= (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

EX $\vec{u} = (1, 2, 3)$, $\vec{v} = (6, 4, 5)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= (2\times5 - 4\times3)\vec{i} - (1\times5 - 6\times3)\vec{j} + (1\times4 - 2\times6)\vec{k}$$

$$= -2\vec{i} + 13\vec{j} - 8\vec{k}$$

$$= (-2, 13, -8)$$

Q WHY DO WE CARE?

A The cross product can be used to

(a) Find a vector $\perp$ to both $\vec{u}$ and $\vec{v}$

(b) Test if $\vec{u} \parallel \vec{v}$

(c) Calculate area of triangle, parallelogram

(d) Calculate volume of parallelipiped.

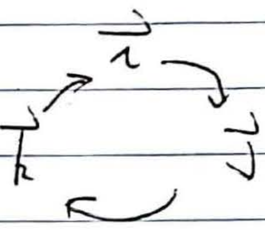
PROPERTIES

① $\vec{u}\times\vec{u} = \vec{0}$ ← ZERO VECTOR!

② $\vec{v}\times\vec{w} = -\,\vec{u}\times\vec{v}$.

③ $\vec{i}\times\vec{j} = \vec{k}$

$\vec{j}\times\vec{k} = \vec{i}$



"GO AROUND CIRCLE"

$\vec{k}\times\vec{i} = \vec{j}$

④ $\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

PF

② $\vec{v}\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$

$$= (v_2u_3 - u_2v_3)\vec{i} + \ldots$$

$$= -\left[(u_2v_3 - v_2u_3)\vec{i} + \ldots\right.$$

$$= -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\,\vec{u}\times\vec{v}$$

③ $\vec{i}\times\vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 1\vec{k}$

$= \vec{k}$.

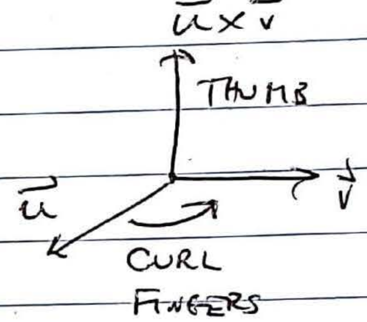
THM [PHYSICS DEFN OF CROSS PRODUCT]

(A) MAGNITUDE OF $\vec{u} \times \vec{v}$

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$$

ASIDE $0 \leq \theta \leq \pi \Rightarrow \sin\theta \geq 0.$

(B) DIRECTION OF $\vec{u} \times \vec{v}$



$\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}$ and $\vec{v}$

AND

$\vec{u} \times \vec{v}$ points in direction given by RIGHT HAND RULE

PF

(A) $|\vec{u} \times \vec{v}|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2$

$$\overset{\text{ALGEBRA}}{=} |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$

$$\underset{\text{DEFN DOT}}{\overset{\text{PHYSICS}}{=}} |\vec{u}|^2 |\vec{v}|^2 \left[1 - \cos^2\theta\right]$$

$$= \left[|\vec{u}|\,|\vec{v}| \sin\theta\right]^2.$$

(B) $\vec{u} \cdot (\vec{u} \times \vec{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \cdots = 0.$

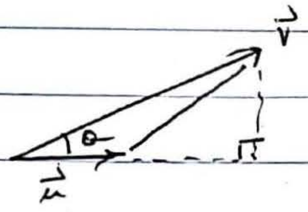
Right Hand Rule is consistent with $\vec{i} \times \vec{j} = \vec{k}$

NOTE IF $|\vec{u}| = |\vec{v}| = 1$ THEN $|\vec{u} \times \vec{v}| = \sin\theta$

CROSS PRODUCT IS $\sin\theta$ IN DISGUISE.

APPLICATIONS

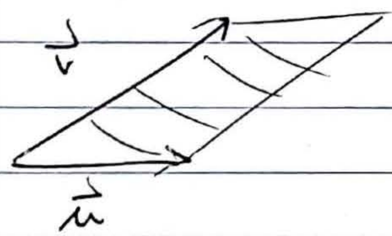
① AREA OF TRIANGLE



$b = |\vec u|$
$h = |\vec v| \sin\theta$

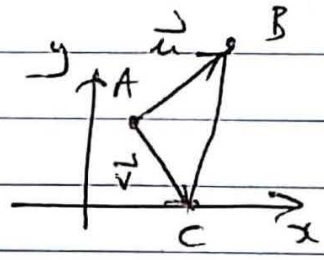
$$A = \frac{1}{2} bh = \frac{1}{2} |\vec u|\, |\vec v| \sin\theta = \frac{1}{2} |\vec u \times \vec v|$$

② AREA OF // GRAM (2 x Area Δ)



$A = |\vec u \times \vec v|$

EX Find Area of Δ with vertices
$A = (1,2)$, $B = (3,4)$, $C = (2,0)$



$\vec u = \overrightarrow{AB} = (2,2)$
$\vec v = \overrightarrow{AC} = (1,-2)$

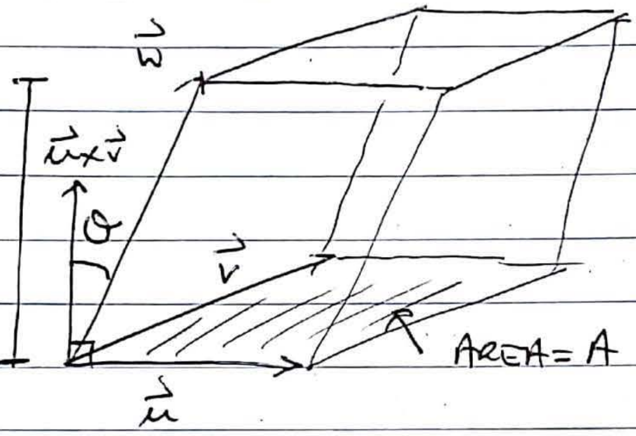
$$A = \frac{1}{2} |\vec u \times \vec v| = \frac{1}{2} . 6 = 3$$

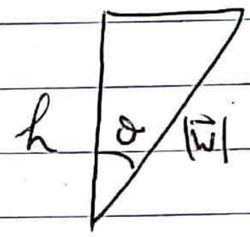
as

$$\vec u \times \vec v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 2 & 2 & 0 \\ 1 & -2 & 0 \end{vmatrix} = -6\vec k$$

NOTE Since $\vec u, \vec v$ are in xy-plane. $\vec u \times \vec v$ is $\perp$ to xy plane. So $\vec u \times \vec v = c\vec k$ for some $c \in \mathbb{R}$

③ VOLUME OF PARALLELEPIPED (SLANTED BOX)





θ = Angle between $\vec{u}\times\vec{v}$ and $\vec{w}$.

$$V = Ah = |\vec{u}\times\vec{v}|\,.\,(|\vec{w}||\cos\theta|)$$

So $$\boxed{V = |(\vec{u}\times\vec{v})\cdot\vec{w}|}$$

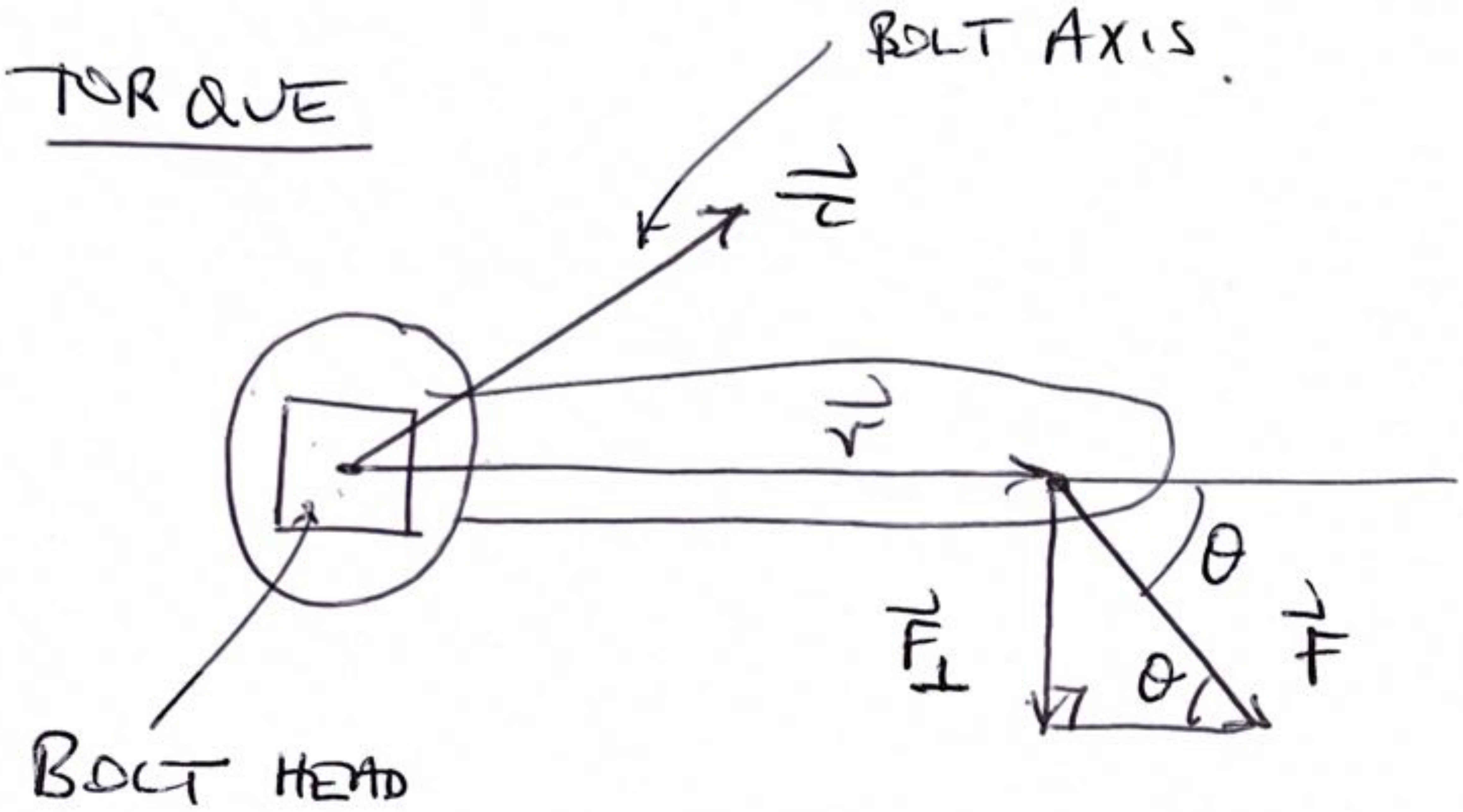
EX If $\vec{u} = (1,2,3)$, $\vec{v} = (6,5,4)$, $\vec{w} = (1,0,7)$

are vectors which determine 3 sides of // piped then

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = \left|\begin{vmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 1 & 0 & 7 \end{vmatrix}\right|$$

$$= |1(5.7 - 0.4) - 2(6.7 - 4.1) + 3(6.0 - 5.1)|$$
$$= |35 - 76 + 18|$$
$$= 23$$

4 TORQUE



Torque, $\vec{\tau}$, measures capability of Force F to produce change in rotational motion of a body

- To loosen a bolt apply force $\vec{F}$ at point $\vec{r}$ on spanner.
- It is only the component $\vec{F}_\perp$ of $\vec{F}$ $\perp$ to $\vec{r}$ that causes bolt to rotate.

(THINK : $\theta = 0 \Rightarrow$ NO ROTATION)

So $\quad |\vec{F}_\perp| = |\vec{F}| \sin\theta$

- The larger $|\vec{r}|$ the larger the torque.
- SUGGESTS $\quad |\vec{\tau}| = |\vec{F}|\,|\vec{r}| \sin\theta$

Set $\quad$ Dir^n of $\vec{\tau}$ = Axis of rotation.

HENCE $\quad \boxed{\vec{\tau} = \vec{r} \times \vec{F}}$