Math 2415

Problem Section #8

Your TA will set aside 40 minutes fro your group to do TWO of the three parts of the Active Learning Models Project #5: Parametrized Surfaces.

For the rest of the session work on the following problems.

14.6: Gradients and Directional Derivatives

1. Let $f(x, y) = e^y \cos(2x)$ and let $\mathbf{x}_0 = (\pi/3, 0)$.

   (a) Find the gradient of f.

   (b) Evaluate the gradient of f at $\mathbf{x}_0$, i.e., find $\nabla f(\mathbf{x}_0)$.

   (c) Find the directional derivative of f at $\mathbf{x}_0$ in the direction $\phi = 3\pi/4$ (i.e, northwest)

   (d) Find the directional derivative of f at $\mathbf{x}_0$ in the direction of the vector $\mathbf{v} = (3, 4)$.

   (e) Find the maximum rate of change of f at $\mathbf{x}_0$ and the direction in which it occurs.

   (f) In what direction is the minimum (i.e. most negative) rate of change of f at $\mathbf{x}_0$?

   (g) Use the gradient of f to calculate the equation of the tangent line to the level curve $f(x, y) = \frac{1}{2}$ at $\mathbf{x}_0$. **Hint:** The gradient of f at $\mathbf{x}_0$ is a vector that is perpendicular to the tangent line. Use this information to find a vector that is parallel to the tangent line.

14.7A: Local Optimization

A guided example

Consider the function $z = f(x, y) = x^2 + 3y^2 - 2xy - 3x$. The critical points of this function are the points in the xy-plane that satisfy the equations

$$0 = \frac{\partial f}{\partial x} = 2x - 2y - 3 \tag{1}$$

$$0 = \frac{\partial f}{\partial y} = 6y - 2x. \tag{2}$$

Equations (1) and (2) are the equations of a pair of lines in the xy-plane. The critical points are the points that lie on **both** of these lines (since both equations need to hold at a critical point). By sketching the two lines you can see that in this example there can only be one critical point of f.

In general the critical point equations

$$0 = \frac{\partial f}{\partial x}(x, y) \tag{3}$$

$$0 = \frac{\partial f}{\partial y}(x, y) \tag{4}$$

are the equations of a **pair of curves** in the xy-plane. The critical points are the points that lie on both curves. Often you can sketch these two curves and use your sketch to determine how many critical points there are and roughly where they are located. Then you can use this geometric information to guide an algebraic calculation to solve Equations (3) and (4) and thereby find the precise location of the critical points. **Use this combined geometric-algebraic technique in the problems below.**

Problems

1. Find the local maxima, minima, and saddle points of the following functions

   (a) $f(x, y) = x^4 + y^2 + 2xy$

   (b) $f(x, y) = y^3 - 3y + 3x^2 y$

   (c) $f(x, y) = xy + 2x + 2y - x^2 - y^2$

   (d) $f(x, y) = xy - x^3 y - xy^3$

   (e) $f(x, y) = x^3 + y^3 - 3x^2 - 6y^2$

   Instructions:

   - For each example plot the curve in the (x, y)-plane where $f_x = 0$ and the curve where $f_y = 0$.
   - Use these curves to locate the critical points (they are the points that lie on both curves).
   - In particular, you should be easily able to see from your picture how many critical points there are and roughly where they are located.
   - Then solve the problem completely algebraically.
   - Double check that the critical points you found really solve the equations $f_x = 0$ and $f_y = 0$.
   - Make sure the picture and the algebra tell the same story! For example, did the algebraic method produce the same number of critical points as in the picture? Are their locations roughly correct?
   - Once you have found the critical points apply the second derivative test to classify them.