Math 2415

Problem Section #5

Make sure you do some problems from each section.

Your TA will set aside 45 minutes for your group to do the Active Learning Models Project #2: Saddle Surfaces. Although the questions in the models project may look different from problems on the homework and exams, they are actually very closely related. The project is designed to increase your understanding and enable you to more readily solve homework and exam problems. For the rest of the session work through the following problems.

13.1: Curves as Interscetions of Surfaces

1. We can find a parametrization of the curve obtained by intersecting the cone $z = \sqrt{x^2 + y^2}$ and the plane $y = 1 + x$ as follows:

   (a) Parametrize the line $y = 1 + x$ in the xy-plane.

   (b) Now that you know how x and y depend on t, use $z = \sqrt{x^2 + y^2}$ to get a formula for z in terms of t.

   (c) In general, what do you need to know about the algebraic forms of the two surfaces in order to use an approach like this? In particular, try a similar approach for the curves obtained by intersecting the pairs of surfaces:

      i. $z = x^2 + y^2$ and $x + y = 2$.

      ii. $x^2 + 2y^2 + z^2 = 4$ and $x = z^2$.

      iii. $x^2 + y^2 = 4$ and $z = xy$ [**Hint**: This one is a bit different: Start by parametrizing the circle $x^2 + y^2 = 4$ in the xy-plane.]

      iv. $z = \sqrt{x^2 + y^2}$ and $z = y + 3$. **Hint:** This time set the right hand sides of the two equations to be equal. That will give you an equation in x and y that represents the shadow of our curve on the xy-plane. Solve that equation for y in terms of x. Then proceed as before.

   If you have extra time, try sketching these curves.

13.2 & 13.3: Calculus of Curves

1. This question asks you to review your understanding of the geometric meaning of the derivative of a parametrized curve.

   (a) Make a sketch of the curve $(x, y) = \mathbf{r}(t) = (t^2, t)$ for $0 \leq t \leq 2$.

   (b) Add the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$ to your sketch. [Don't worry about getting the points exactly correct!]

   (c) Calculate the tangent vector, $\mathbf{r}'(1)$, and add it to your sketch.

   (d) Calculate the secant vector
   $$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$
   add it to your sketch, and explain why it is almost the same as $\mathbf{r}'(1)$.

2. Let C be the curve in $\mathbb{R}^2$ parametrized by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$, for $-2 \le t \le 2$.
   (a) Show that the curve C is given by $x^2 = y^3$.
   (b) Sketch the curve C. **Hint:** C has a cusp at the origin.
   (c) Calculate $\mathbf{r}'(0)$.
   (d) Suppose now that $\mathbf{r}$ is the position of a particle moving in the plane as a function of time. Describe the motion of the particle, especially near $t = 0$.
   (e) Calculate the speed and the acceleration vector of the particle at time $t = 1$.

3. Let C be the curve $(x, y, z) = \mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$ for $0 \le t \le 2$.
   (a) Parametrize the tangent line to C at $t = 1$ in such a way that the motion along the tangent line is a good approximation to the motion along the curve near $t = 1$.
   (b) Set up a one-dimensional integral for the length of C.
   (c) **[Trick]** Show that the function under the square root in the integrand is a perfect square, that is it can be expressed in the form $[g(t) + h(t)]^2$ for some functions g and h.
   (d) Hence, evaluate the integral.
   (e) If we changed the curve to $(x, y, z) = \mathbf{r}(t) = \sqrt{3}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$ could you still do the problem? Why? *Hint:* Nope!!

14.1, Functions of Several Variable

1. Let $z = f(x, y) = e^{-x-y}$. Sketch the contours of $f(x, y) = k$ for $k = 0.5$, $k = 1$, $k = 2$. Use this information to help you sketch the graph of f.

2. Sketch a contour map for the function $z = f(x, y) = y^{1/3} + x$. *Hint:* Solve $y^{1/3} + x = k$ for y in terms of x.

3. 14.1.61

4. 14.1.62

5. You could also try 14.1.63–14.1.66 if you like!