Math 2415

Problem Section #3

Make sure you do some problems from each section.

12.5B: Planes

Recall the following definitions:

(i) A **vector parametrization** of the line through the endpoint of the vector $\mathbf{a}$ in the direction of the vector $\mathbf{b}$ is given by $\ell(t) = \mathbf{a} + t\mathbf{b}$, where $t \in \mathbf{R}$.

(ii) A **scalar parametrization** of the line in (i) is

$$\begin{aligned} x &= a_1 + tb_1 \\ y &= a_2 + tb_2 \\ z &= a_3 + tb_3 \end{aligned}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

(iii) A **level set equation** of a plane is an equation of the form $ax + by + cz = d$, where a, b, c, d are real numbers.

(iv) A **parametrization** of a plane through the endpoint of the vector $\mathbf{u}$ that contains the vectors $\mathbf{v}$ and $\mathbf{w}$ is of the form $\mathbf{r}(s, t) = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where $s, t \in \mathbf{R}$.

For each problem start by drawing a schematic diagram that illustrates the geometrical relationships between the various points, lines, vectors, planes in the problem. Use your diagram to help you set up equations that will help you solve the problem.

1. This problem concerns the plane through the point $Q = (2, 0, 3)$ that is perpendicular to the line with parameterization $\ell(t) = \mathbf{p} + t\mathbf{v} = (1 + 3t)\mathbf{i} + (2 - t)\mathbf{j} + 4t\mathbf{k}$.

   (a) Draw a schematic diagram showing the relationship between the plane and the line. Include the point, Q, and the vectors, $\mathbf{p}$ and $\mathbf{v}$ as well as the normal vector to the plane in your sketch.

   (b) Use you sketch to help find a level set equation and *two* different parameterizations for the plane.

2. (a) With the aid of a schematic diagram explain why there is a plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$. Explain why there is only one such plane. Find a level set equation of this plane.

   (b) Is there a plane that contains the line in (2a) and is parallel to the plane $5x+2y+2z = 1$? Explain.

3. Find the three intercepts of the plane $3x + y + 2z = 12$ and use them to sketch the plane. How would you work out the area of the triangle in the plane that is formed by these three intercepts? (Don't do the calculation: Just explain.)

4. Find a parametrization of the plane that contains both the point $(2, 4, 6)$ and the line $x = 7 - 3t$, $y = 3 + 4t$, $z = 5 + 2t$. Make sure you start by constructing a schematic diagram!
5. Consider the plane through the point $P = (-1, 3, 6)$ with normal vector $\mathbf{n} = (5, 2, -1)$.
   (a) Find a level set equation for this plane.
   (b) Convert your equation to the graph of a function of the form $z = f(x, y)$
   (c) Finally convert your graph equation to a parameterization of the plane and check that it contains the point P and has normal $\mathbf{n}$.
6. Consider the line $\ell(t) = (1 + 2t, -1 - t, 3t)$. Find the point of intersection of this line with the plane $x - 2y - 3z = 8$. Does this line intersect the y-axis?
7. Find a parametrization for the line of intersection of the planes $3x - 6y - 2z = 3$ and $2x + y - 2z = 2$.
8. **[Extra]** In most cases, the intersection of two lines in $\mathbf{R}^2$ is a point. In most cases, what can you say about the following situations:
   (a) The intersection of two lines in $\mathbf{R}^3$
   (b) The intersection of two planes in $\mathbf{R}^3$
   (c) The intersection of a line and a plane in $\mathbf{R}^3$

15.7&8A: Cylindrical and Spherical Coordinates

1. Plot the point with cylindrical coordinates, $(r, \theta, z) = (2, 4\pi/3, 1)$. Then find the rectangular and spherical coordinates of the point.
2. Let P be the point whose rectangular coordinates are $(x, y, z) = (-1, -1, 2)$ Find the cylindrical and spherical coordinates of P.
3. Plot the point with spherical coordinates, $(\rho, \theta, \phi) = (3, \pi/3, \pi/6)$. Then find the rectangular and cylindrical coordinates of the point.
4. Write the equation $z = x^2 + y^2$ in cylindrical and in spherical coordinates.
5. Write the equation $r = 2$ in rectangular and in spherical coordinates.
6. Write the equation $\phi = \pi/2$ in rectangular and in cylindrical coordinates.
7. Sketch the following solids, surfaces, curves, and points, *altogether in one plot, with labels*. For each surface, convert the equation to rectangular and to cylindrical coordinates. For each point, find the cylindrical and rectangular coordinates of that point.
   (a) The surface $\rho = 4$.
   (b) The curve where $\rho = 4$ and $\theta = \pi/2$.
   (c) The curve where $\rho = 4$ and $\phi = \pi/3$.
   (d) The point $(\rho, \theta, \phi) = (4, \pi/2, \pi/3)$.
   (e) The solid where $\rho \leq 4$, $0 \leq \theta \leq \pi/2$ and $\pi/3 \leq \phi \leq \pi/2$. (Use a separate plot for this solid.)