Math 2415

Problem Section #2

Make sure you do some problems from each section.

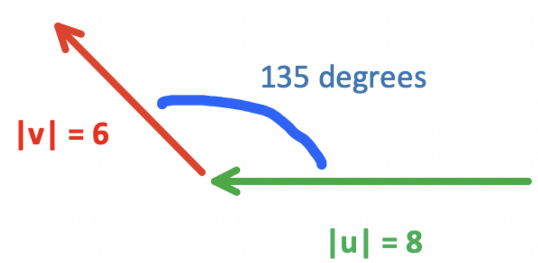
12.3: Projections

1. Let $P = (1, 2, 3)$, $Q = (4, 0, -5)$ and $R = (4, 7, 9)$. Calculate the scalar and vector projections of the vector $\overrightarrow{PR}$ onto the vector $\overrightarrow{PQ}$.

2. (a) Let $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (2, 4, -1)$.
   (b) Find the scalar and vector projections of $\mathbf{u}$ onto $\mathbf{v}$.
   (c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree (use a calculator!)
   (d) Find three nonzero vectors that are orthogonal to $\mathbf{u}$.

3. Answer this problem using the picture below. You are *not* allowed to calculate the components of the vectors $\mathbf{u}$ and $\mathbf{v}$. *Warning:* Look carefully at the directions of the arrows on the vectors. Relate to theory from lectures!

   (a) Find $\mathbf{u} \cdot \mathbf{v}$
   (b) Use triangle geometry to find the scalar projection of $\mathbf{v}$ onto $\mathbf{u}$.
   (c) Use triangle geometry to find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$. (Write your answer in terms of $\mathbf{v}$.)
   (d) What are the scalar and vector projections of $\mathbf{v}$ onto $\mathbf{u}$?



4. **[Challenge]** Let $\mathbf{a} = 2\mathbf{i} + 4\mathbf{k}$.

   (a) Find one vector $\mathbf{b}$ so that $\text{Comp}_\mathbf{a}(\mathbf{b}) = 2$.
   (b) Draw a picture to convince yourself that there are an infinite number of vectors, $\mathbf{b}$ for which $\text{Comp}_\mathbf{a}(\mathbf{b}) = 2$. Describe this set of vectors using an English sentence.

12.4: Cross Products

For Questions 3-7 below you must draw a schematic diagram (sketch, picture) that shows the relationships between the various points, vectors, planes etc in the problem before attempting to solve it algebraically. Do not make these pictures entirely realistic. For example, if you have a point $p = (1, 2, 3)$ in a plane P don't draw it exactly. Just draw any old point in any old plane.

1. Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. Calculate $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2. Let $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j}$.

   (a) Use the following properties of cross products (rather than the formula for the cross product in terms of a determinant) to calculate $\mathbf{a} \times \mathbf{b}$:

      i. the linearity property $\mathbf{u} \times (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \times \mathbf{v} + b\mathbf{u} \times \mathbf{w}$, for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and all scalars, a, b, and c

      ii. the formulae $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, and

      iii. the anti-symmmetry property $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$, for all vectors $\mathbf{u}$ and $\mathbf{v}$

   (b) Using your answer to (a) find the volume of the parallelipiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

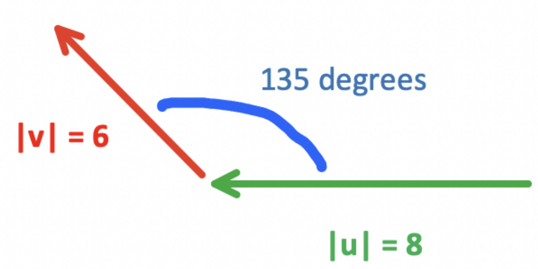
   (c) Now find the volume of this parallelipiped by computing a 3×3 determinant. Hopefully, you got the same answer as in (b)?

3. Answer this problem using the picture below. You are *not* allowed to calculate the components of the vectors $\mathbf{u}$ and $\mathbf{v}$. *Warning:* Look carefully at the directions of the arrows on the vectors. Relate to theory from lectures!

   (a) Find $|\mathbf{v} \times \mathbf{u}|$

   (b) Determine whether $\mathbf{v} \times \mathbf{u}$ is directed in or out of the page

   (c) Calculate the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.



4. Find two unit vectors orthogonal to both $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (1, 0, -2)$.

5. Let $A = (1, 2, 3)$, $B = (0, 1, -1)$, $C = (2, 4, -3)$, and $D = (1, 0, 2)$

   (a) Find a vector orthogonal to the plane that contains the points A, B, and C.

   (b) Find the area of the triangle ABC.

(c) Find the volume of the parallelepiped three of whose edges are given by AB, AC, and AD.

6. *Note: This problem is related to a problem on PH2.*

   (a) Explain why there is at least one vector $\mathbf{v}$ so that $(1,2,4)\times\mathbf{v}=(2,-3,1)$.

   (b) Is there a vector $\mathbf{v}$ so that $(1,2,4)\times\mathbf{v}=(2,3,-1)$?

7. Find examples of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ so that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\neq(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$.

8. Find all vectors $\mathbf{v}$ so that $\mathbf{i}\times\mathbf{v}=\mathbf{k}$. **Hint:** Algebraically express $\mathbf{v}$ as $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and work out what the condition $\mathbf{i}\times\mathbf{v}=\mathbf{k}$ tells you about a, b, c. Once you have the answer, draw a picture showing all the vectors, $\mathbf{v}$ that work and convince yourself they really do work.

9. *Note: This problem is related to a problem on PH2.* Suppose that $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ and $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. Use

   (a) the linearity property $\mathbf{u}\times(a\mathbf{v}+b\mathbf{w})=a\mathbf{u}\times\mathbf{v}+b\mathbf{u}\times\mathbf{w}$, for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and all scalars, a, b, and c

   (b) the formulae $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$, and

   (c) the anti-symmmetry property $\mathbf{v}\times\mathbf{u}=-\mathbf{u}\times\mathbf{v}$, for all vectors $\mathbf{u}$ and $\mathbf{v}$

   to derive the familiar formula

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

10. [**Challenge**] What must you know about a pair of nonzero vectors, $\mathbf{a}$ and $\mathbf{b}$, to ensure that there is a vector, $\mathbf{x}$, so that $\mathbf{x}\times\mathbf{a}=\mathbf{b}$ and $\mathbf{x}\cdot\mathbf{a}=|\mathbf{a}|$? In this situation how many such vectors $\mathbf{x}$ are there? Argue both geometrically and algebraically.

12.5A: Lines

Recall the following definitions:

(i) A **vector parametrization** of the line through the endpoint of the vector $\mathbf{a}$ in the direction of the vector $\mathbf{b}$ is given by $\mathbf{r}(t)=\mathbf{a}+t\mathbf{b}$, where $t\in\mathbf{R}$.

(ii) A **scalar parametrization** of the line in (i) is

$$x=a_1+tb_1$$
$$y=a_2+tb_2$$
$$z=a_3+tb_3$$

where $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$.

For each problem start by drawing a schematic diagram that illustrates the geometrical relationships between the various points, lines, and vectors in the problem. Use your diagram to help you set up equations that will help you solve the problem.

1. Find vector and scalar parameterizations of the line through $P = (1, 2, -1)$ and $Q = (-1, 0, 2)$.

2. Find *two* vector parameterizations of the line through the point $(3, 2, 1)$ that is parallel to the line $x = 2 + 3t$, $y = 4t$ $z = -1 + 5t$.

3. Find a vector parameterizations of the line through the point $P = (1, 2, -1)$ that is perpendicular to both the vectors $\mathbf{i} - \mathbf{2}j$ and $\mathbf{j} + 3\mathbf{k}$.