Math 2415

Problem Section #12

Do at least some problems from each section.
Recommended (in this order): 15.7 and 15.8 (1,2,3,4,6), 15.9.1, 15.9.3, 15.9.4
15.6.2a, 15.6.2b, 15.6.3, 15.6.5, 15.3.2, 15.3.4

You will have the opportunity to do the remaining problems next week.

15.3, Double Integrals in Polar Coordinates

1. Evaluate $\iint_D e^{x^2+y^2}\,dA$, where D is the region in the 1st quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
2. Evaluate $\iint_D \cos(x^2+y^2)\,dA$, where D is the region bounded by the semicircle $x = \sqrt{9-y^2}$ and the y-axis.
3. Calculate the volume of the solid under $z = x^2 + y^2$ and above $x^2 + y^2 \leq 16$. Sketch this solid.
4. Calculate the volume of the solid below the plane $x + 2y + 3z = 6$ and above $x^2 + y^2 \leq 1$.
5. Evaluate the integral by converting to polar coordinates: $\int_0^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (x + 2y)\,dy\,dx$. Your answer should be in terms of the parameter, R.

15.6, Triple Integrals in Rectangular Coordinates

1. A solid is often described as the region enclosed by a collection of surfaces. For example, the unit cube is the solid enclosed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$. When we sketch the cube we do not sketch the solid. Neither do we sketch the surfaces that form the boundary of the cube. Rather we sketch the edges of the cube. Try that for yourself! Explain why the edges of the cube are given by the intersections of the pairs of surfaces that form the boundary of the cube.
2. For each part, sketch the region bounded by the given surfaces. As explained above, each pair of the surfaces intersects in a curve. We sketch the solid by sketching these curves. Then use a triple integral to calculate the volume of the solid.
   (a) $x = z^2$, $x = 8 - z^2$, $y = 1$, $y = 3$.
   (b) $z = 10 - x^2 - y^2$, $y = x^2$, $x = y^2$, $z = 0$
   (c) The region in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) that is bounded by the generalized cylinder $y = 1 - x^2$ and the plane $z = 1 - y$.
3. Set up an iterated triple integral to calculate the mass of the solid E that is bounded by $y = z^2$, $y = z$, $x + y + z = 2$, $x = 0$. Assume that the density of the material in the solid is given by $f(x, y, z) = z^2$.
4. Evaluate $\iiint_E y\,dV$, where E is the solid bounded by the surfaces $z = 2 - x^2$, $z = x^2 - 2$, $y = 0$ and $y = 1$.
5. Find the volume of the region in the first octant that is bounded by the planes $x + z = 2$ and $y + 2z = 4$.

15.7 and 15.8, Triple Integrals in Cylindrical and Spherical Coordinates

1. Use cylindrical coordinates to find the volume of the solid that lies outside the cylinder $x^2 + y^2 = 3$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

2. Let E be the solid region in the first octant (*i.e.*, where $x \geq 0$, $y \geq 0$, $z \geq 0$) that is inside the cylinder $x^2 + y^2 = 1$ and below the plane $x + z = 1$. Let $f(x, y, z) = y$ be the density of the material in the solid. Calculate the total mass of the solid.

3. Sketch the solid region, E, in the first octant that is bounded by the cylinder $y^2 + z^2 = 16$ and the plane $x + y = 4$. Use a triple integral in cylindrical coordinates to find $J = \iiint_E \sqrt{y^2 + z^2}\, dV$. Explain why $\frac{J}{\text{Volume}(E)}$ is the average distance of a point in E from the x axis.

4. Evaluate the integral $\iiint_E z\, dV$, where E is the solid bounded by the spheres $x^2+y^2+z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant. If $f(x, y, z) = z$ is the density function of the solid E explain how to use your answer to find the total mass of the solid.

5. Let E be the solid region $x^2 + y^2 + z^2 \leq 16$. Calculate $\iiint_E z^4\, dV$.

6. Use spherical coordinates to calculate the triple integral $\iiint_E z\, dV$, where E is the solid region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

7. Find $\iiint_E \frac{1}{(x^2+y^2+z^2)^{3/2}}\, dV$ where E is the region between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, with $0 < a < b$.

15.9: Change of Variables Theorem

1. Evaluate $\iint_R (x-y)^2 e^{x+y}\, dx dy$ where R is the parallelogram bounded by $x+y = 1$, $x+y = 3$, $x - y = -2$ and $x - y = 1$. **Hint:** Use the Change of Variables Theorem with $u = x + y$ and $v = x - y$.

2. (*Skip this one if you understand Q1.*) Use the Change of Variables Theorem to evaluate the integral $\iint_R y\, dA$, where R is the quadrilateral region bounded by the lines $x + 2y = 2$, $x + 2y = 4$, $x = 0$, and $y = 0$. **Hint:** Let $u = x + 2y$ and $v = y$.

3. Use the change of variables formula and an appropriate transformation to evaluate $\iint_R x\, dA$, where R is the square with vertices $(0, 0)$, $(2, 2)$, $(4, 0)$, and $(2, -2)$.

4. Calcuate $\iint_R y^2\, dA$, where R is the region bounded by the ellipse $4x^2 + 25y^2 = 1$. Hint: Use the change of variables $u = 2x$, $v = 5y$.

5. Let D be the region in the first quadrant of the xy-plane bounded by the curves $y = \frac{x}{2}$, $y = x$, $xy = 4$ and $xy = 9$. Calculate $\iint_D x\, dx dy$. **Hint:** Use the change of variables $x = ve^u$, $y = ve^{-u}$.