Math 2415
Problem Section #1

Make sure you do some problems from each section.

12.1: 3D Coordinate Systems

1. Draw a rectangular box with the origin and the point $(1, 2, 3)$ as opposite vertices and faces parallel to the coordinate planes. Label each vertex with its coordinates. Find the length of the diagonal of the box.

2. (a) What does the equation $x = 2$ represent in $\mathbb{R}^2$? Sketch!

   (b) What does the equation $x = 2$ represent in $\mathbb{R}^3$? Sketch!

   (c) What does the equation $z = 1$ represent in $\mathbb{R}^3$? Sketch!

   (d) Describe the set of all points, (x, y, z), in $\mathbb{R}^3$ for which $x = 2$ and $z = 1$. Sketch!

3. For what values of b and c do the points $(1, 2, 3)$, $(4, 5, 1)$, and $(10, b, c)$ all lie on the same line?

4. (a) Find the equation of the sphere with center $(1, 3, 5)$ and radius 4.

   (b) What is the intersection of this sphere with the xz-plane? Argue algebraically and geometrically.

   (c) What would the radius of the sphere have to be for the the intersection of the sphere and the xz-plane to be a single point. What are the coordinates of this point?

12.2: Vectors

1. Do not use coordinate representations of vectors to solve this problem. Just draw pictures.

   (a) Draw two vectors that are not parallel and label them **a** and **b**.

   (b) Sketch the vector $\mathbf{a} + \mathbf{b}$

   (c) Sketch the vector $\mathbf{a} - \frac{1}{2}\mathbf{b}$

   (d) Sketch the vector $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

2. Sketch a parallelogram and label the vertices A, B, C, and D going around counter-clockwise from the bottom left vertex. Let E be the point obtained by intersecting the two diagonals of the parallelogram. Make sure the side lengths of your parallelogram are not all equal, ie you did not draw a rhombus. The notation $\overrightarrow{AB}$ refers to the displacement vector from the point A to the point B.

   (a) Name all pairs of equal vectors in your sketch.

   (b) Write each combination of vectors as a single vector: $\overrightarrow{AB} + \overrightarrow{BC}$, $\overrightarrow{AE} - \overrightarrow{EB}$, $2\overrightarrow{AB} + \overrightarrow{BD}$.

3. Find a unit vector parallel to $(1, 2, 3)$.

4. Let $\mathbf{a} = 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find

(a) $\mathbf{a} + 2\mathbf{b}$

(b) $|\mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}|$.

5. Suppose that $\mathbf{v} \in \mathbb{R}^2$ lies in the 2nd quadrant, makes an angle of $120°$ with the positive x-axis, and has length $|\mathbf{v}| = 2$. Find the coordinates of $\mathbf{v}$.

### 12.3: The Dot Product

1. Find $\mathbf{a} \cdot \mathbf{b}$ if

    (a) $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (-2, 3)$,

    (b) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$,

    (c) $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $120°$.

2. (a) Let $\mathbf{u} = (3, -2, 1)$ and $\mathbf{v} = (2, 4, -1)$.

    (b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree (use a calculator!)

    (c) Find three nonzero vectors that are orthogonal to $\mathbf{u}$.

3. Find $\mathbf{u} \cdot \mathbf{v}$
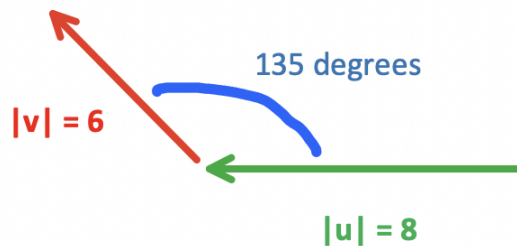
4. Let C be the point on the line segment AB that is twice as far from A and it is from B, and let O denote the origin. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$. [Recall that $\overrightarrow{AB}$ refers to the displacement vector from the point A to the point B.]

    (a) Make a sketch showing the relationships between all these points and vectors. Your sketch will help you solve the other parts of the problem.

    (b) Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

    (c) Hence express $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

    (d) Hence find a formula for $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

5. What restrictions must be made on b so that the vector $2\mathbf{i} + b\mathbf{j}$ is orthogonal to (a) $-3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and (b) $\mathbf{k}$?

6. The cube in the figure has side length one with one vertex at the origin, O.

    (a) Let Q be the vertex opposite O and let R be the midpoint of a face of the cube as shown in the figure. Find a formula for the vectors $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ in terms of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

(b) Find the angle between the vectors **i** and $\overrightarrow{OQ}$.

(c) Find the angle between the vectors $\overrightarrow{OQ}$ and $\overrightarrow{OR}$.
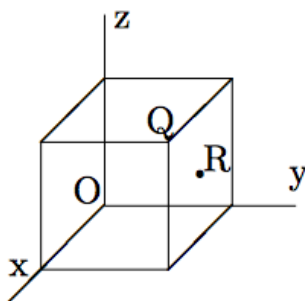
Extra Challenge Questions:

1. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

2. We learned in class that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$. Use this property to prove the Cauchy-Schwarz inequality

$$|\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}||\mathbf{b}|. \tag{1}$$

3. The triangle inequality states that

$$|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|. \tag{2}$$

   (a) Give a geometric interpretation of the triangle inequality.
   (b) Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and the distribute law for the dot product to prove (2).

4. The parallelogram law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2. \tag{3}$$

   (a) Give a geometric interpretation of the parallelogram law.
   (b) Prove the parallelogram law.