LAST NAME:	FIRST NAME:
POISSON	SIMÉON DENIS

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MATH 4362, Spring 2024

Midterm Exam One

(Zweck)

Instructions: This 75 minute exam is worth 75 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

(1) [15 pts] Find all solutions $u = f(r)$ of the three-dimensional Laplace equation, $u_{xx} + u_{yy} + u_{zz} = 0$, that depend only on the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$. You may use (without proof) the fact that in spherical coordinates (r, θ, ϕ), the Laplacian operator is given by

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left\{ \frac{1}{\sin^2 \phi} f_{\theta\theta} + f_{\phi\phi} + \cot \phi f_\phi \right\}.$$

$$0 = u_{rr} + \frac{2}{r} u_r$$

Let $v = u_r$

So
$$0 = v_r + \frac{2}{r} v$$

$$\int \frac{dv}{v} = -\int \frac{2}{r} dr$$

$$\log |v| = -2 \log r + C_1$$

$$v(r) = \frac{C_2}{r^2}$$

$$u = \int v \, dr = \int \frac{C_2}{r^2} dr$$

$$u = \frac{C_3}{r} + C_4$$ where $C_1 - C_4$ are all constants

UPSHOT

$$u(x, y, z) = \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}} + \beta$$

for some constants α, β.

(2) [15 pts] Solve the initial value problem for $u = u(t, x)$ given by

$$u_t - 4u_x + 2u - 1 = 0,$$
$$u(0, x) = \frac{1}{1 + x^2}.$$

Let $v(t, x) = u(t, x - 4t)$ ①

So $u(t, x) = v(t, x + 4t)$ ②

Then $v_t = u_t - 4u_x = 1 - 2u = 1 - 2v$

So $\int \frac{dv}{1-2v} = \int dt$

$$-\frac{1}{2} \ln |1 - 2v| = t + C_1$$

$$\ln |1 - 2v| = -2t + C_2$$

$$1 - 2v = A e^{-2t} \quad A \in \mathbb{R}, ~~constant~~$$

So $\boxed{v(t, x) = \frac{1}{2}\left(1 - A(x) e^{-2t}\right)}$

Find $A(x)$ using

$$\frac{1}{1+x^2} = u(0, x) = v(0, x) = \frac{1}{2}(1 - A(x))$$

So $A(x) = 1 - \frac{2}{1+x^2}$

By ②

$$u(t, x) = v(t, x + 4t) = \frac{1}{2}\left[1 - \left(1 - \frac{2}{1 + (x+4t)^2}\right) e^{-2t}\right]$$

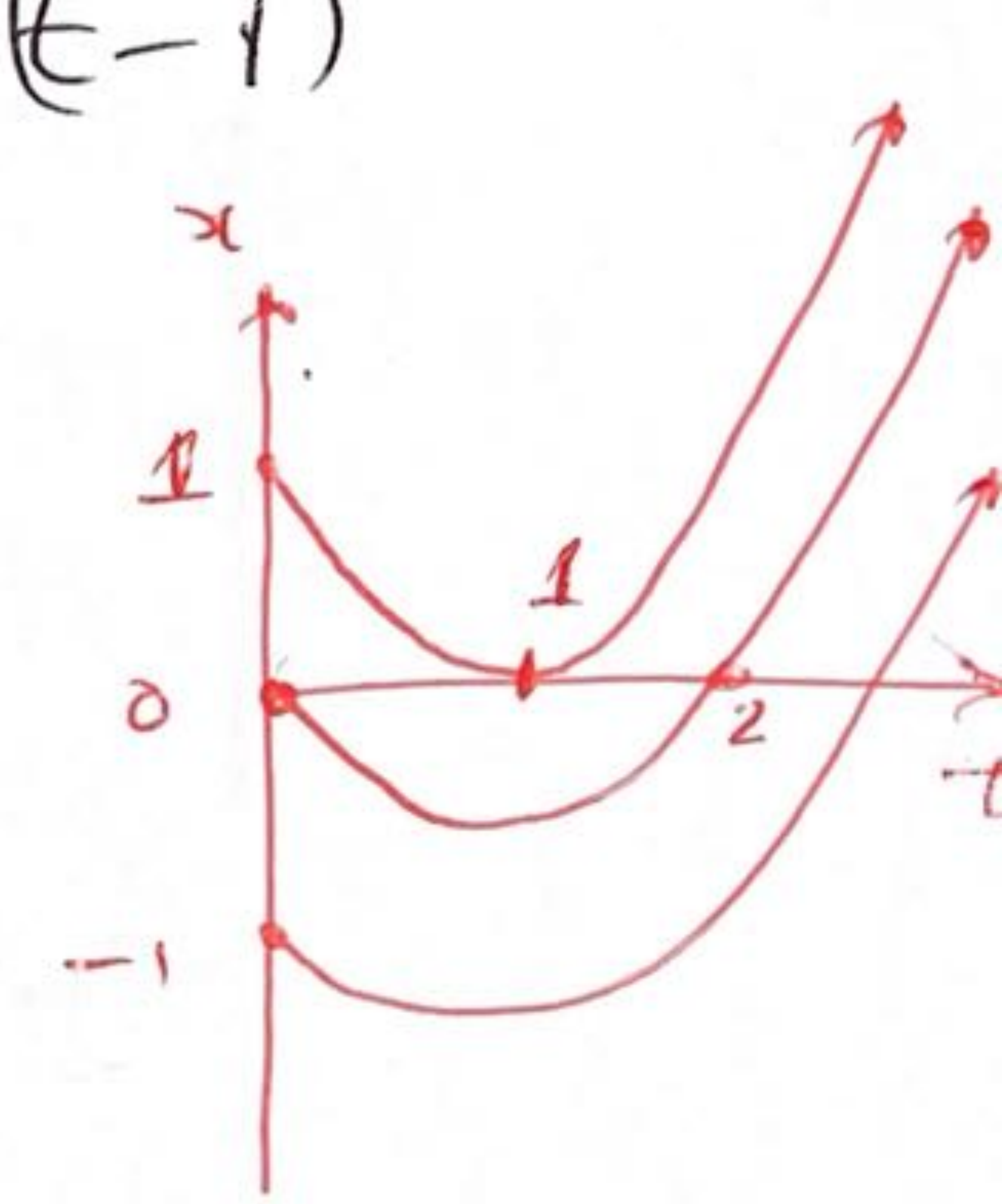
(3) [15 pts] Solve the initial value problem for $u = u(t, x)$ given by

$$u_t + 2(t-1)u_x = 0,$$
$$u(0, x) = e^{-x^2}.$$

In addition, sketch the characteristic curves which go through the points $(0, x_0)$ for $x_0 = -1, 0, 1$. Finally, sketch the solution $u = u(t, x)$ at $t = 1$ and $t = 4$.

$$c(t,x) = 2(t-1)$$

Char Curves: $\begin{cases} \dfrac{dx}{dt} = c(t,x) = 2(t-1) \\ x(0) = x_0 \end{cases}$



So $x(t) = x_0 + \int_0^t 2(s-1)\,ds$

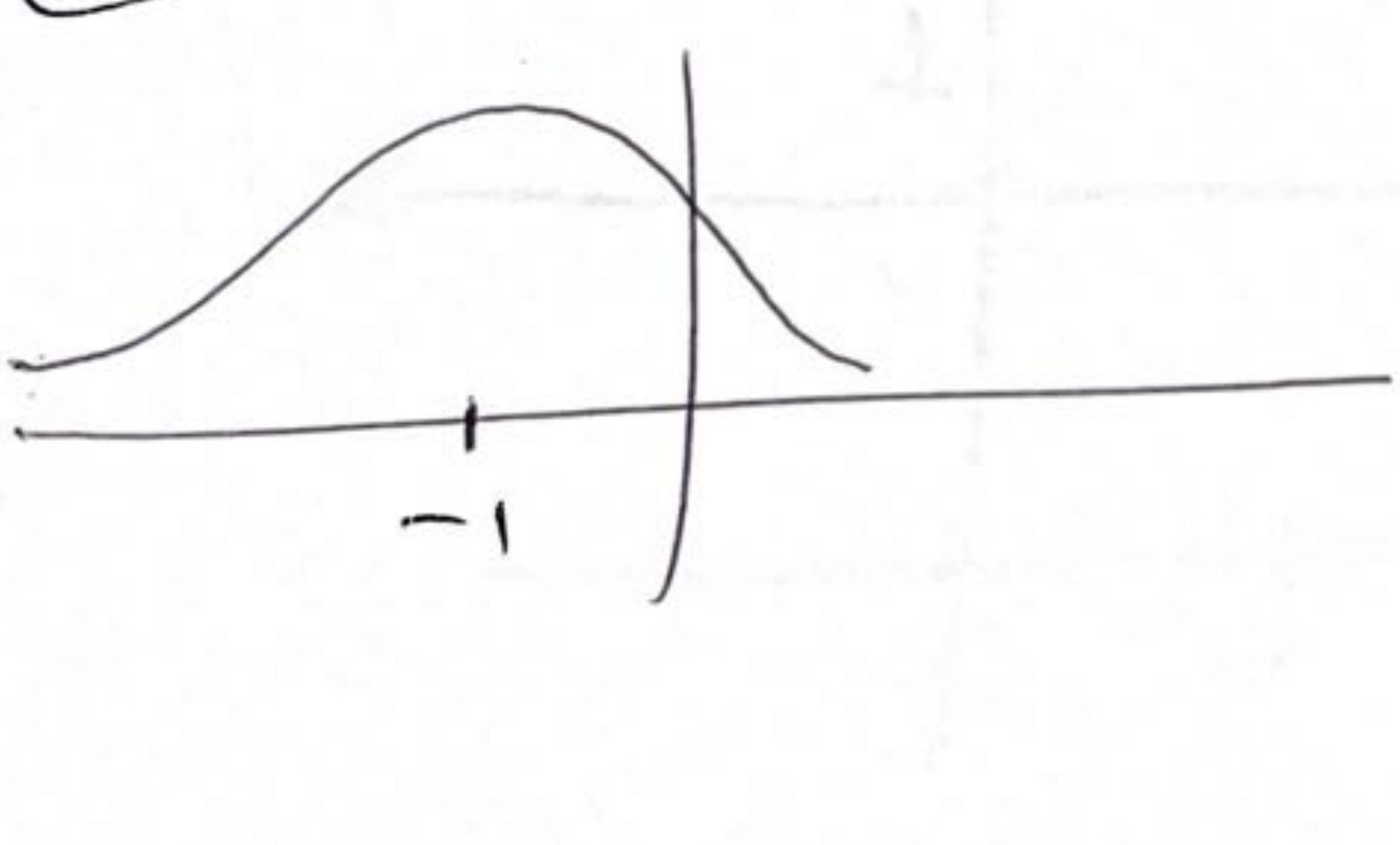
$$x(t) = x_0 + \left[(s-1)^2\right]_{s=0}^{s=t}$$

$$\boxed{x(t) = x_0 + (t-1)^2 - 1}$$

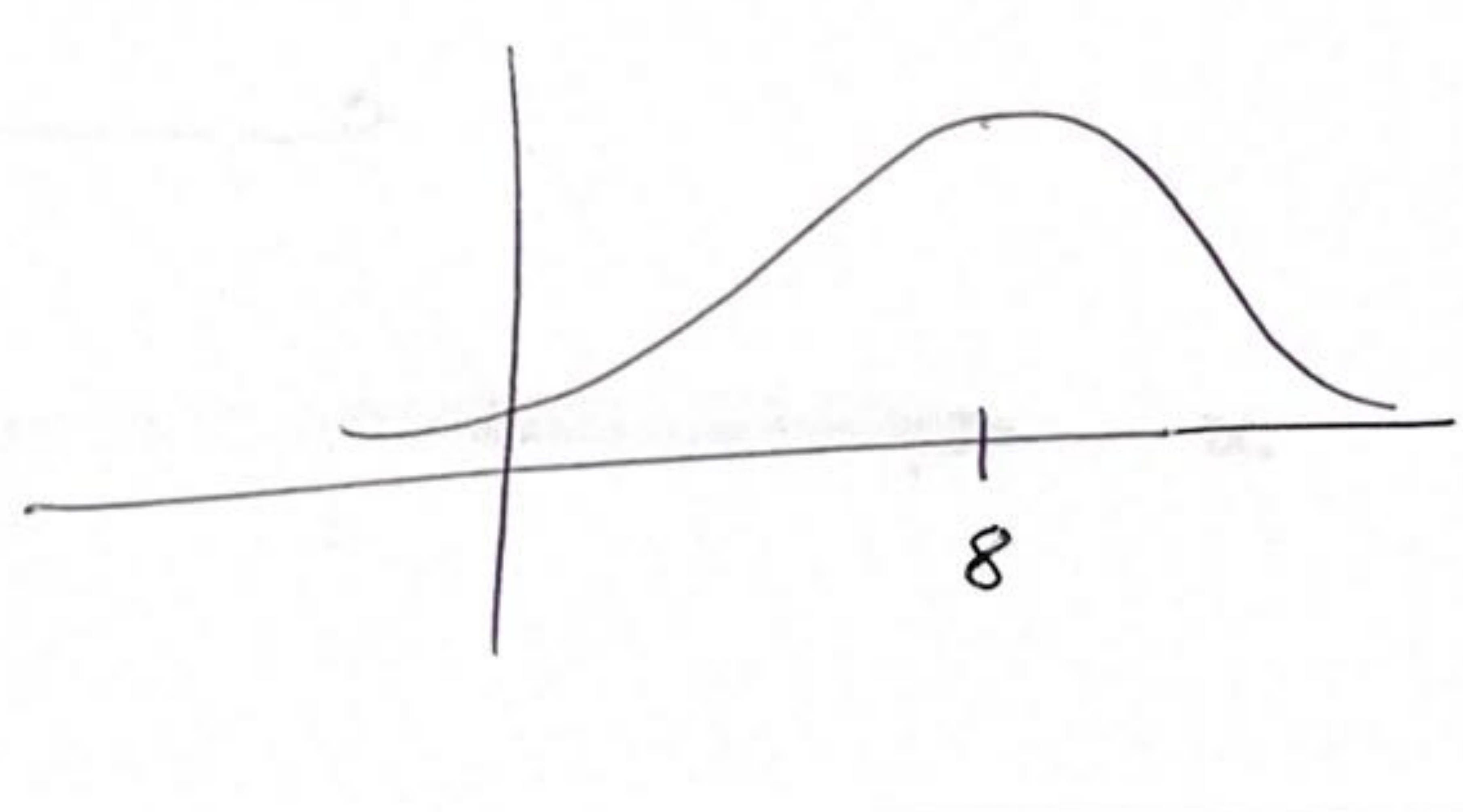
NOW u is constant along CCs. So

$$u(t,x) = u(0, x_0) = e^{-x_0^2} = e^{-[x+1-(t-1)^2]^2}$$

$\boxed{t=1}$



$\boxed{t=4}$ $\quad 1-(t-1)^2 = 1-9 = -8$



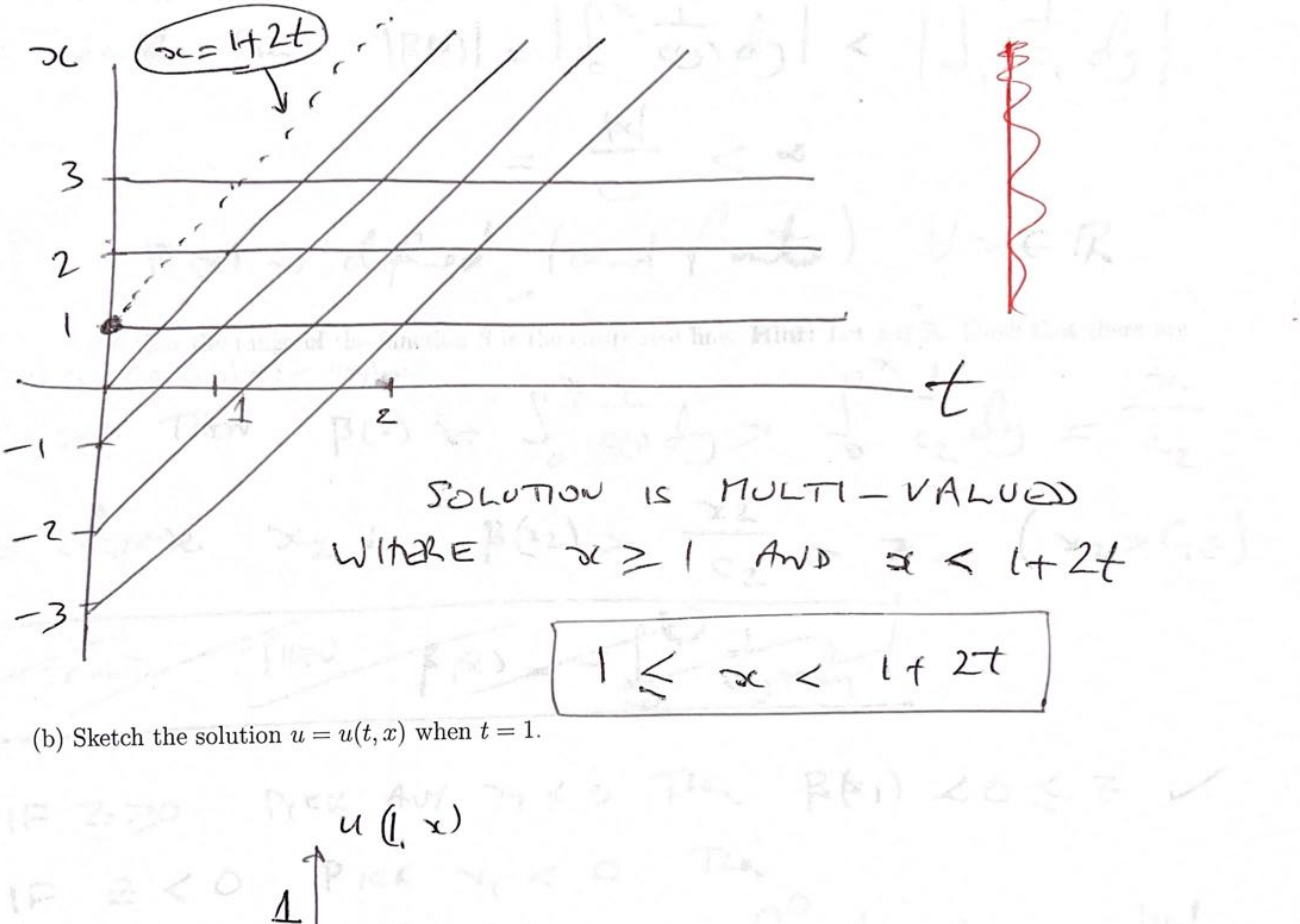
(4) [15 pts] Consider the initial value problem

$$u_t + 2uu_x = 0, \tag{1}$$

$$u(0,x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x \geq 1. \end{cases} \tag{2}$$

(a) Sketch the characteristic curves in the (t,x)-plane through the points $(0,x_0)$ for $x_0 = -3, -2, -1, 0, 1, 2, 3$. Identify the region of the (t,x)-plane where the solution $u = u(t,x)$ is a multi-valued function.

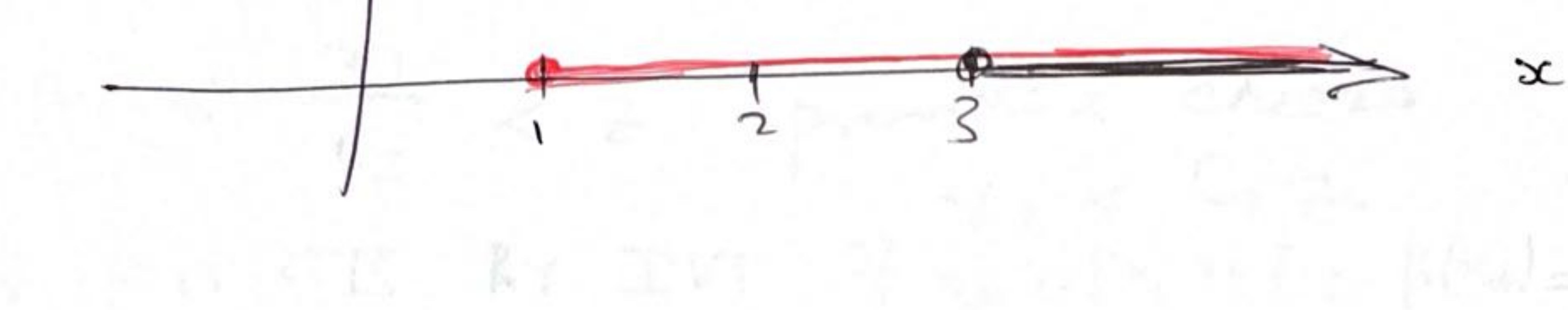
SLOPE OF C.L. = $2 \times$ VALUE OF u as $\frac{dx}{dt} = 2u(t,x)$



SOLUTION IS MULTI-VALUED WHERE $x \geq 1$ AND $x < 1+2t$

$$\boxed{1 \leq x < 1+2t}$$

(b) Sketch the solution $u = u(t,x)$ when $t = 1$.



(5) [15 pts] Consider the initial value problem

$$u_t + c(x)u_x = 0, \tag{3}$$
$$u(0,x) = f(x), \tag{4}$$

where the function $c = c(x)$ is continuous and satisfies the condition that there exist constants C_1 and C_2 so that

$$0 < C_1 < c(x) < C_2 \qquad \text{for all } x \in \mathbb{R}.$$

Let $\beta(x) = \int_0^x \frac{1}{c(y)}\,dy$.

(a) Show that the domain of the function β is the entire real line.

LET $x \in \mathbb{R}$. Then $|\beta(x)| = \left|\int_0^x \frac{1}{c(y)}\,dy\right| < \left|\int_0^x \frac{1}{c_1}\,dy\right|$

$= \frac{|x|}{c_1} < \infty$

So $\beta(x)$ is defined (and finite) $\forall x \in \mathbb{R}$

(b) Show that the range of the function β is the entire real line. **Hint:** Let $z \in \mathbb{R}$. Show that there are x_1, x_2 so that $\beta(x_1) < z < \beta(x_2)$.

$\boxed{\text{FIX } z}$

(a) IF $x > 0$ THEN $\beta(x) = \int_0^x \frac{1}{c(y)}\,dy > \int_0^x \frac{1}{c_2}\,dy = \frac{x}{c_2}$

So choose x_2 : $\beta(x_2) > \frac{x_2}{c_2} > z$ $\quad (x_2 > C_2 z)$

~~(b) IF $x < 0$ THEN $\beta(x) = -\int_x^0 \frac{1}{c(y)}\,dy$~~

(b) IF $z \geq 0$ PICK ANY $x_1 < 0$ Then $\beta(x_1) < 0 \leq z$ ✓

IF $z < 0$ PICK $x_1 < 0$. Then

$\beta(x_1) = -\int_{x_1}^0 \frac{dy}{c(y)} < -\int_{x_1}^0 \frac{1}{c_2}\,dy = -\frac{|x_1|}{c_2}$

$\beta(x_1) < \frac{x_1}{c_2} < z$ provided choose $x_2 < C_2 z$.

(c) SINCE β IS CTS BY IVT $\exists x_* \in [x_1, x_2] : \beta(x_*) = z$

(c) Explain why the function $\beta : \mathbb{R} \to \mathbb{R}$ is invertible.

$$\beta(x) = \int_0^x \frac{1}{c(y)}\,dy \Rightarrow \beta'(x) = \frac{1}{c(x)} > 0$$

SO β IS INCREASING + HENCE β IS INVERTIBLE

(d) Show that the characteristic curve through (t_*, x_*) intersects the x-axis at the point $(0, x_0)$ where $\beta(x_0) = \beta(x_*) - t_*$.

$$\frac{dx}{dt} = c(x) \quad \text{WITH} \quad \begin{matrix} x(t_*) = x_* \\ x(0) = x_0. \end{matrix}$$

HAS $\displaystyle\int_{x_0}^{x_*} \frac{dx}{c(x)} = \int_0^{t_*} dt$ OR $\beta(x_*) - \beta(x_0) = t_*$

So $\beta(x_0) = \beta(x_*) - t_*$

For the rest of this problem we consider the special case that $c(x) = 1 - \frac{3}{4+x^2}$.

(e) Show that the characteristic curve, $x = x(t)$, through $(0, x_0)$ satisfies the equation

$$x + 3\arctan(x) = t + x_0 + 3\arctan(x_0).$$

$$c(x) = 1 - \frac{3}{4+x^2} = \frac{1+x^2}{4+x^2}$$

$$\frac{1}{c(x)} = \frac{4+x^2}{1+x^2} = 1 + \frac{3}{1+x^2}$$

$$\beta(x) - \beta(x_0) = \int_{x_0}^{x} \frac{1}{c(y)}\,dy = \int_{x_0}^{x} \left(1 + \frac{3}{1+y^2}\right) dy$$

$$= x - x_0 + 3\arctan(x) - 3\arctan(x_0)$$

So by (d) get

$$x + 3\arctan(x) - x_0 - 3\arctan(x_0) = t.$$

[More space for answer to (e)]

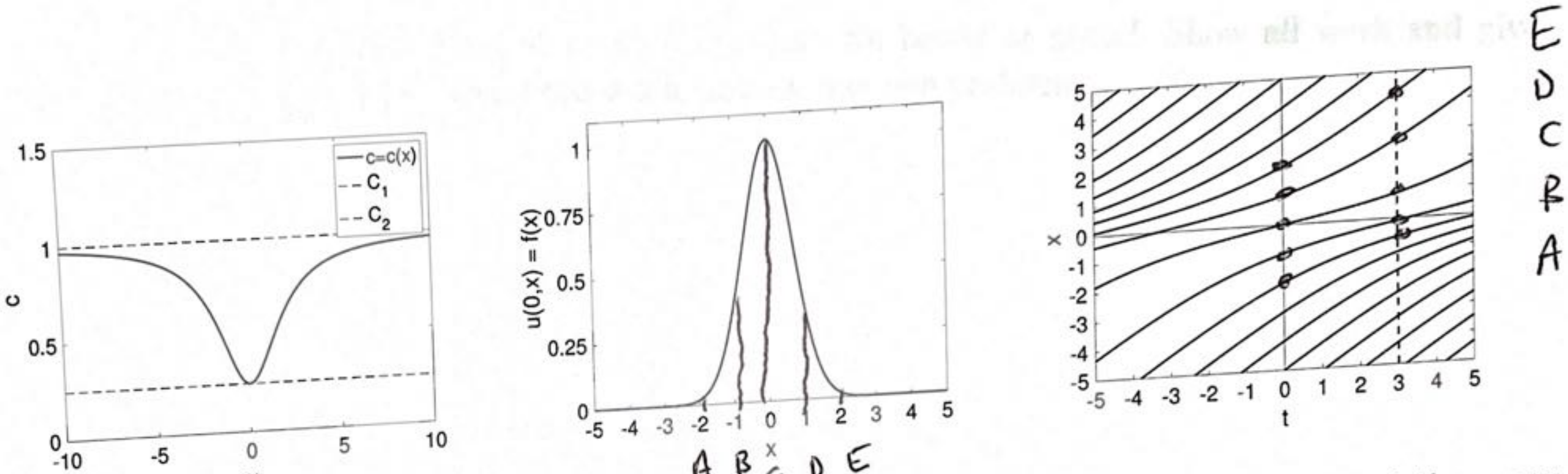


Figure 1: Left: The graph of c. Middle: Graph of Gaussian initial condition. Right: Characteristic curves with integer values of x_0.

(f) Sketch the solution $u = u(t, x)$ at time $t = 3$ in the special case that $u(0, x) = f(x) = e^{-x^2}$.