LAST NAME:	FIRST NAME:

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MATH 4362, Spring 2024

Midterm Exam One

(Zweck)

Instructions: This 75 minute exam is worth 75 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

(1) [15 pts] Find all solutions $u = f(r)$ of the three-dimensional Laplace equation, $u_{xx} + u_{yy} + u_{zz} = 0$, that depend only on the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$. You may use (without proof) the fact that in spherical coordinates (r, θ, ϕ), the Laplacian operator is given by

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left\{ \frac{1}{\sin^2 \phi} f_{\theta\theta} + f_{\phi\phi} + \cot \phi f_\phi \right\}.$$

(2) [15 pts] Solve the initial value problem for $u = u(t, x)$ given by

$$\begin{aligned} u_t - 4u_x + 2u - 1 &= 0, \\ u(0, x) &= \frac{1}{1 + x^2}. \end{aligned}$$

(3) [15 pts] Solve the initial value problem for $u = u(t, x)$ given by

$$u_t + 2(t-1)u_x = 0,$$
$$u(0, x) = e^{-x^2}.$$

In addition, sketch the characteristic curves which go through the points $(0, x_0)$ for $x_0 = -1, 0, 1$. Finally, sketch the solution $u = u(t, x)$ at $t = 1$ and $t = 4$.

(4) [15 pts] Consider the initial value problem

$$u_t + 2uu_x = 0, \tag{1}$$

$$u(0,x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x \geq 1. \end{cases} \tag{2}$$

(a) Sketch the characteristic curves in the (t,x)-plane through the points $(0,x_0)$ for $x_0 = -3,-2,-1,0,1,2,3$. Identify the region of the (t,x)-plane where the solution $u = u(t,x)$ is a multi-valued function.

(b) Sketch the solution $u = u(t,x)$ when $t = 1$.

(5) [15 pts] Consider the initial value problem

$$u_t + c(x)u_x = 0, \tag{3}$$
$$u(0, x) = f(x), \tag{4}$$

where the function $c = c(x)$ is continuous and satisfies the condition that there exist constants C_1 and C_2 so that

$$0 < \; C_1 \; < \; c(x) \; < C_2 \qquad \text{for all } x \in \mathbb{R}.$$

Let $\beta(x) = \int_0^x \frac{1}{c(y)}\, dy$.

(a) Show that the domain of the function β is the entire real line.

(b) Show that the range of the function β is the entire real line. **Hint:** Let $z \in \mathbb{R}$. Show that there are x_1, x_2 so that $\beta(x_1) < z < \beta(x_2)$.

(c) Explain why the function $\beta : \mathbb{R} \to \mathbb{R}$ is invertible.

(d) Show that the characteristic curve through (t_*, x_*) intersects the x-axis at the point $(0, x_0)$ where $\beta(x_0) = \beta(x_*) - t_*$.

For the rest of this problem we consider the special case that $c(x) = 1 - \frac{3}{4+x^2}$.

(e) Show that the characteristic curve, $x = x(t)$, through $(0, x_0)$ satisfies the equation

$$x + 3\arctan(x) = t + x_0 + 3\arctan(x_0).$$

[More space for answer to (e)]

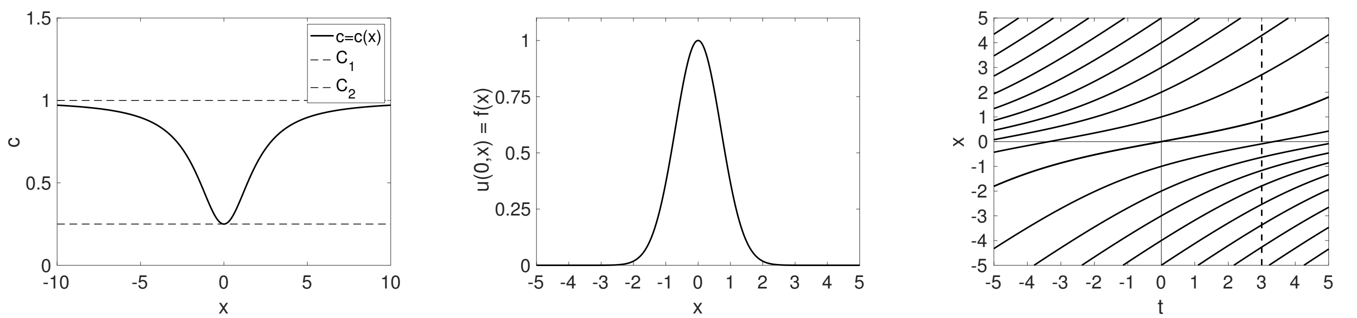

Figure 1: Left: The graph of c. Middle: Graph of Gaussian initial condition. Right: Characteristic curves with integer values of x_0.

(f) Sketch the solution $u = u(t, x)$ at time $t = 3$ in the special case that $u(0, x) = f(x) = e^{-x^2}$.