LAST NAME: SOLUTIONS	FIRST NAME:

1 /12	2 /12	3 /10	4 /10	5 /10
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MATH 4362 (Spring 2018), Final Exam, (Zweck)

Instructions: This 2 hour 45 minute exam is worth 100 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

Throughout this exam we define

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

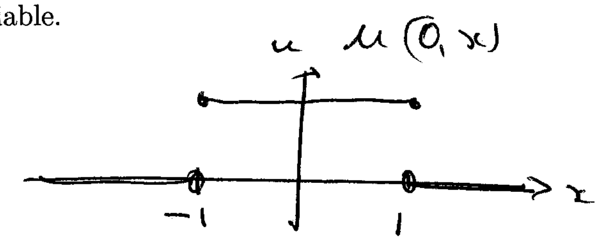
(1) [12 pts] **True or false? Give brief explanations for your answers.**

(a) Suppose that $u = u(t,x)$ solves

$$u_t + u_x = 0 \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R},$$
$$u(0,x) = \chi_{[-1,1]}(x) \qquad \text{for } x \in \mathbb{R}.$$

Then the function $v(x) = u(1,x)$ is differentiable.

<u>FALSE</u>



The initial condition is not continuous at $x = \pm 1$

The solution to $u_t + u_x = 0$ is $u(t,x) = f(x-t)$ where $f(x) = u(0,x)$. So v is not differentiable as a function of x, since its is not continuous.

(b) Suppose that $u = u(t, x)$ solves

$$u_t = u_{xx} \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R},$$
$$u(0, x) = \chi_{[-1,1]}(x) \qquad \text{for } x \in \mathbb{R}.$$

Then the function $v(x) = u(1, x)$ is differentiable.

The heat equation has the property that no matter how unsmooth the initial condition is, the solution at any time $t > 0$ is infinitely differentiable as a function of x

So v is differentiable

$\boxed{\text{TRUE}}$

(c) Suppose that $u = u(t, x)$ solves

WAVE EQN

$$u_{tt} = u_{xx} \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R},$$
$$u(0, x) = \chi_{[-1,1]}(x) \qquad \text{for } x \in \mathbb{R},$$
$$u_t(0, x) = 0 \qquad \text{for } x \in \mathbb{R}.$$

Let $w(t) = u(t, 2)$. Then there is a time $T > 0$ so that $w(t) = 0$ for all $0 < t < T$.

By d'Alembert's formula the solution is

$$u(t,x) = \frac{1}{2}\left[\chi_{[-1,1]}(x-t) + \chi_{[-1,1]}(x+t)\right]$$

$$w(t) = \frac{1}{2}\left[\chi_{[-1,1]}(2-t) + \chi_{[-1,1]}(2+t)\right]$$

$$= \frac{1}{2}\left[\chi_{[1,3]}(t) + \chi_{[-3,-1]}(t)\right]$$

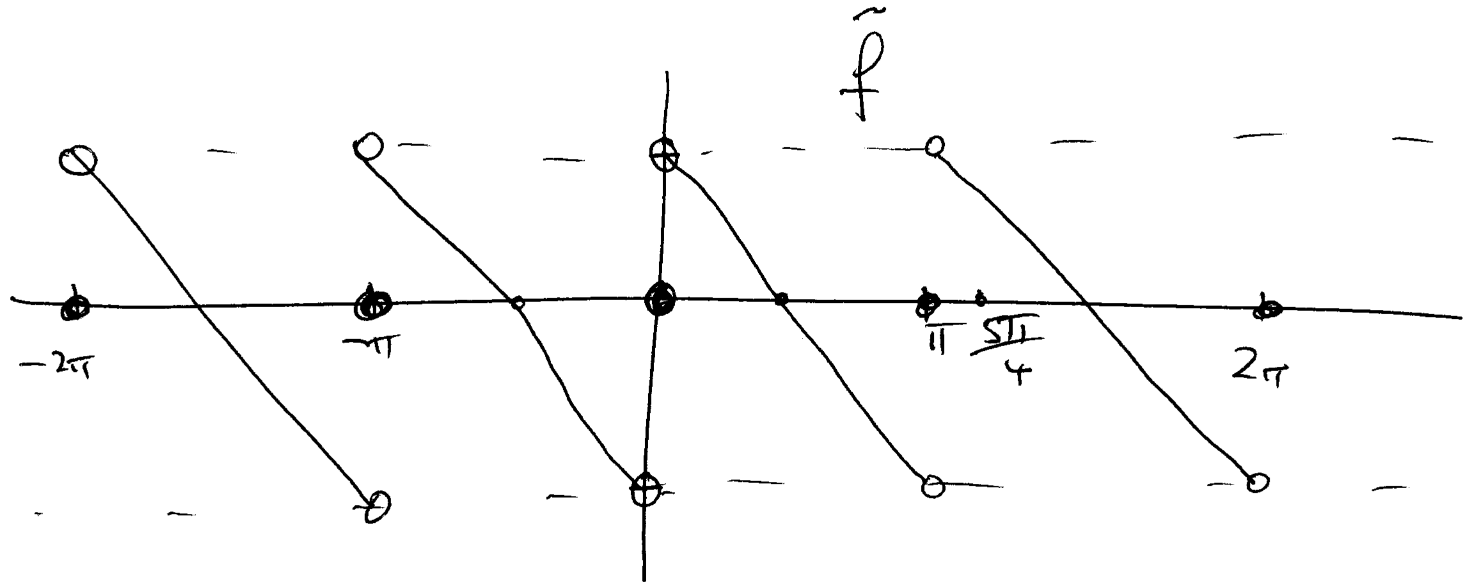
So $T = 1$ works

$\boxed{\text{TRUE}}$

BIG IDEA

Waves propogate information (encoded in initial conditions) at finite speed.

(2) [12 pts] Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \frac{\pi}{2} - x$ for $x \in (0, \pi)$ and $f(\pi) = 0$. Let $\widetilde{f}$ be the 2π-periodic odd extension of f. Graph $\widetilde{f}$. Explain why the Fourier series of $\widetilde{f}$ converges pointwise but not uniformly on $\mathbb{R}$. What is the value of the Fourier series of $\widetilde{f}$ at (i) $x = 0$ and (ii) $x = \frac{5\pi}{4}$?



$\hat{f}$ is piecewise continuous on $\mathbb{R}$. So its Fourier Series converges pointwise to $\tilde{f}$.

However if $\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin nx$ converges uniformly then since partial sums are continuous then infinite series would be too. But this series equals $\tilde{f}$, which is NOT continuous. So F.S. does not converge uniformly.

$(\text{F.S. } \tilde{f} \text{ @ } x=0) = \frac{1}{2}[\hat{f}(0^+) + \hat{f}(0^-)] = \frac{1}{2}[\frac{\pi}{2} - \frac{\pi}{2}] = 0$

$(\text{F.S. of } \hat{f} \text{ @ } x = \frac{5\pi}{4}) = \hat{f}(5\pi/4) = \pi/4$

$\uparrow$ $\hat{f}$ cts @ $5\pi/4$

(3) [10 pts] Solve for $u = u(t,x)$ on $t \geq 0$ and $x \in \mathbb{R}$:

$$\begin{aligned} u_{tt} &= 4u_{xx} + \underset{\substack{\| \\ F(t,x)}}{\sin t}, \\ u(0,x) &= e^{-x^2}, = f(x) \\ u_t(0,x) &= 0. \end{aligned}$$

$c = 2$

$$u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{s=0}^{s=t}\int_{y=x-c(t-s)}^{y=x+c(t-s)} F(s,y)\,dy\,ds$$

$$= \frac{1}{2}\left[e^{-(x-2t)^2} + e^{-(x+2t)^2}\right] + \frac{1}{4}\int_{s=0}^{t}\int_{y=x-(t-s)}^{y=x+(t-s)} \sin(s)\,dy\,ds$$

$$= I_1 + I_2$$

$$I_2 = \frac{1}{4}\int_{s=0}^{t} \sin(s)\left\{[x+(t-s)] - [x-(t-s)]\right\} ds$$

$$= \frac{1}{4}\int_{s=0}^{t} 4(t-s)\sin(s)\,ds \qquad \begin{aligned} u &= t-s & v' &= \sin s \\ u' &= -1 & v &= -\cos s \end{aligned}$$

$$= \left[-(t-s)\cos s\right]_{s=0}^{s=t} - \int_{s=0}^{s=t} (-1)(-\cos s)\,ds$$

$$= t - \sin t$$

So

$$u(t,x) = \frac{1}{2}\left[e^{-(x-2t)^2} + e^{-(x+2t)^2}\right] + t - \sin t$$

(4) [10 pts] Prove that for each $t > 0$ the series,

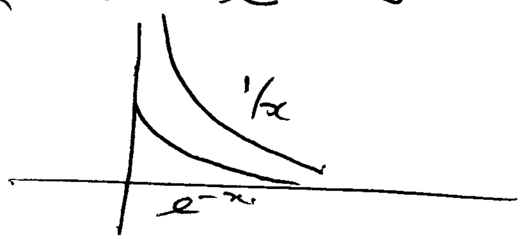
$$u(t,x) \;=\; \sum_{k=0}^{\infty} e^{-k^2 t}\cos(kx),$$

converges uniformly for $x \in \mathbb{R}$. Hence show that u is continuous where $t > 0$.
Hint: Fix $\epsilon > 0$. Prove that for all $t > \epsilon$ the series converges uniformly for $x \in \mathbb{R}$.

Fix $\varepsilon > 0$

Let $t > \varepsilon$

Then $e^{-k^2 t} < e^{-k^2 \varepsilon} < \frac{1}{\varepsilon}\frac{1}{k^2}$ for large enough k as $e^{-x} < \frac{1}{x}$ for x large enough



So $\left|e^{-k^2 t}\cos kx\right| \le \frac{1}{\varepsilon}\frac{1}{k^2} =: a_k$

Since $\sum_{k=1}^{\infty} a_k$ converges, by Weierstrass M-Test for each $t > 0$ $u(t,x)$ converges uniformly on $x \in \mathbb{R}$.

Since each term $e^{-k^2 t}\cos kx$ is continuous we conclude that for each t, $u(t,x)$ is continuous.

(5) [10 pts] Suppose that $u = u(t, x)$ solves

$$\begin{aligned} u_t &= u_{xx}, && \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) &= \arctan(x). \end{aligned}$$

Let $v(t, x) = u_t(t, x)$. Show that v solves

$$\begin{aligned} v_t &= v_{xx}, && \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ v(0,x) &= \frac{-2x}{(1+x^2)^2}. \end{aligned}$$

(a) $v = u_t$

$v_t = u_{tt} = (u_{xx})_t = (u_t)_{xx} = v_{xx}$.

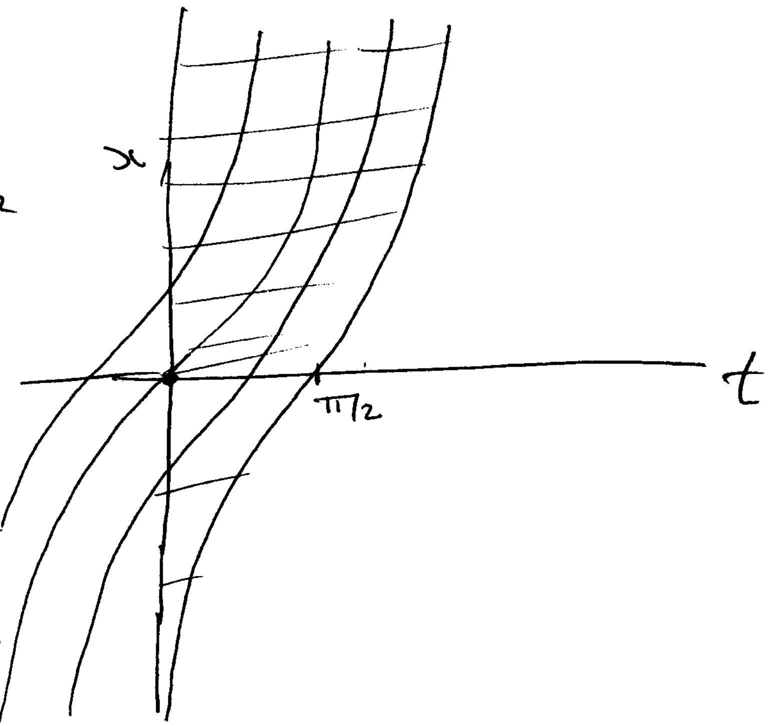
(b) $v = u_t = u_{xx}$

So $v(0, x) = u_{xx}(0, x) = \frac{d^2}{dx^2}(\arctan x)$

$= \frac{-2x}{(1+x^2)^2}$

(6) [12 pts] Suppose that $u = u(t,x)$ solves $u_t + (1+x^2)u_x = 0$. Find and sketch the characteristic curves. Shade that portion of the (t,x)-plane with $t > 0$ where the solution is determined by the values of u at $t = 0$. Derive a formula for the solution, $u = u(t,x)$, with initial values $u(0,x) = f(x)$.

$c(x) = 1 + x^2$



C.C. $\frac{dx}{dt} = c(x) = 1 + x^2$

$\int \frac{dx}{1+x^2} = \int dt$

$\arctan x = t + C$

$\boxed{x(t) = \tan(t+c)}$

C. Curve thru (t_1, x_1) has

$x_1 = \tan(t_1 + c)$

$c = \arctan x_1 - t_1$

So $x(t) = \tan(t - t_1 + \arctan x_1)$

Since u is constant along CCs

$u(t_1, x_1) = u(t_1, x(t_1)) = u(0, x(0))$

$= f(x(0)) = f(\tan(\arctan(x_1) - t_1))$

or $u(t,x) = f(\tan(\arctan(x) - t))$.

(7) [8 pts] Let $h(x,y) = \chi_{[-\pi/4,\pi/4]}(\theta)$ where $(x,y) = (\cos\theta, \sin\theta)$. Let $u = u(x,y)$ solve Laplace's equation

$$\begin{aligned} \Delta u = 0 \qquad & \text{in } x^2+y^2<1, \\ u = h \qquad & \text{on } x^2+y^2=1. \end{aligned}$$

True or false? Give brief explanations for your answers. In the following, $u = u(r,\theta)$.

(a) $u(0,0) < u(1,0)$ (b) $u(0,0) < u(1,\pi)$ (c) $u(0.9,\pi) < u(0.9,0)$.

Hint: The solution is given in polar coordinates by

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,K(r,\theta-\phi)\,d\phi, \qquad \text{where } K(r,\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

(a) $u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,d\phi$ as $K(0,\theta) = 1$

$= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} 1\,d\phi = \frac{1}{4}$

$u(1,0) = h(0)$ by boundary data.

$= 1$

So $u(0,0) < u(1,0)$ $\boxed{\text{TRUE}}$

(b) $u(1,\pi) = h(\pi) = 0$. $u(0,0) = \frac{1}{4}$ $\boxed{\text{FALSE}}$

(c) $u(0.9,0) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} K(0.9,-\phi) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{1-0.9^2}{1+0.9^2-1.8\cos\phi}\,d\phi$

$u(0.9,\pi) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} K(0.9,\pi-\phi) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{1-0.9^2}{1+0.9^2+1.8\cos\phi}\,d\phi$

as $\cos(\pi-\phi) = -\cos\phi$.

Now $1+0.9^2+1.8\cos\phi > 1+0.9^2-\cos\phi$ on $[-\pi/4, \pi/4]$ as $\cos\phi > 0$ there

So $u(0.9,0) < u(0.9,\pi)$. $\boxed{\text{TRUE}}$

(8) [12 pts] Let $C_0^\infty(\mathbb{R})$ be the space of infinitely differentiable functions, $u : \mathbb{R} \to \mathbb{R}$, with the property that there exists an $R > 0$ so that $u(x) = 0$ for all $|x| > R$.

(a) Define what it means for $L : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ to be a distribution.

For each $u \in C_0^\infty(\mathbb{R})$, $L(u) \in \mathbb{R}$ and

L is Linear in that

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

$$\forall c_1, c_2 \in \mathbb{R}, \quad \forall u_1, u_2 \in C_0^\infty(\mathbb{R})$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function and $u \in C_0^\infty(\mathbb{R})$. Show that $L_g(u) = \int_{-\infty}^{\infty} g(x)u(x)\,dx$ is a distribution. LET $u \in C_0^\infty(\mathbb{R})$

We know $\exists R > 0 : |u(x)| = 0$ for $|x| > R$, AND $\exists M_2 : |u(x)| \leq M_2 \ \forall x$

Also g PWCTS $\Rightarrow$ g bounded on $[-R, R]$ by some M_1

So $$|L_g(u)| \leq \int_{-R}^{R} |g(x)||u(x)|\,dx \leq M_1 M_2 \cdot 2R < \infty.$$

AND $$L_g(c_1 u_1 + c_2 u_2) = \int_{\mathbb{R}} g(c_1 u_1 + c_2 u_2)\,dx$$
$$= c_1 \int g u_1\,dx + c_2 \int g u_2\,dx$$
$$= c_1 L_g(u_1) + c_2 L_g(u_2)$$

(c) Let $\xi \in \mathbb{R}$. Define the Dirac delta distribution, δ_ξ, at $x = \xi$, and show that δ_ξ is indeed a distribution.

$$\delta_\xi(u) = u(\xi) \quad \text{for } u \in C_0^\infty(\mathbb{R})$$

$|\delta_\xi(u)| = |u(\xi)| < \infty$ ✓ $\quad \delta_\xi(u) \in \mathbb{R} \quad \forall u \in C_0^\infty(\mathbb{R})$

$$\delta_\xi(c_1 u_1 + c_2 u_2) = (c_1 u_1 + c_2 u_2)(\xi)$$
$$= c_1 u_1(\xi) + c_2 u_2(\xi)$$
$$= c_1 \delta_\xi(u_1) + c_2 \delta_\xi(u_2) \Rightarrow \delta_\xi \text{ LINEAR}$$

(d) Let σ_ξ be the piecewise continuous function defined by

$$\sigma_\xi(x) = \begin{cases} 0 & \text{if } x \leq \xi \\ 1 & \text{if } x > \xi. \end{cases}$$

Show that the derivative of the distribution L_{σ_ξ} equals the Dirac distribution, δ_ξ.

$$\left(L_{\sigma_\xi}\right)'(u) = -L_{\sigma_\xi}(u')$$

$$= -\int_{-\infty}^{\infty} \sigma_\xi(x)\, u'(x)\, dx$$

$$= -\int_{\xi}^{\infty} u'(x)\, dx$$

$$\overset{FTC}{=} -\big[\underbrace{u(\infty)}_{=0 \text{ as } u \in C_0^\infty} - u(\xi)\big]$$

$$= u(\xi)$$

$$= \delta_\xi(u)$$

So $\left(L_{\sigma_\xi}\right)' = \delta_\xi$

Soln $u(t,x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^n}{\pi(2n+1)} e^{-(2n+1)^2 t} \cos[(2n+1)t]$

(9) [14 pts] Find a Fourier series solution, $u = u(t,x)$, for $t > 0$ and $x \in [0,\pi]$, of

$$\begin{aligned} u_t &= u_{xx}, \\ u_x(t,0) &= 0, \\ u_x(t,\pi) &= 0, \\ u(0,x) &= \chi_{[0,\pi/2]}(x). \end{aligned}$$

NEUMANN BCs.

You may assume the eigenvalues satisfy $\lambda \geq 0$.

Separable Solutions $u(t,x) = w(t)\, v(x)$

$$\frac{w'}{w} = -\lambda = \frac{v''}{v}$$

$w = e^{-\lambda t}$. $\lambda > 0$

$v'' + \lambda v = 0$ Write $\lambda = \omega^2$

$v'' + \omega^2 v = 0$ $v' = -A\omega \sin \omega x + B\omega \cos \omega x$

$\boxed{v(x) = A\cos(\omega x) + B\sin(\omega x)}$ ~~$v' = -A\omega \sin \omega x + B\omega \sin(\omega x)$~~

$0 = v'(0) = -B\omega \Rightarrow B = 0$ or $\omega = 0$.

$\boxed{\omega = 0 \text{ gives } u = 1}$

$\boxed{B = 0}$ Gives $0 = v'(\pi) = A\cos(\pi\omega) \Rightarrow \omega = k \in \mathbb{N}$
$\lambda = k^2$

So $\boxed{u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-k^2 t} \cos kx}$

$a_0 = \frac{2}{\pi}\int_0^{\pi} \chi_{[0,\pi/2]}(x)\, dx = 1$

$a_k = \frac{2}{\pi}\int_0^{\pi} \chi_{[0,\pi/2]}(x) \cos kx\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos kx\, dx$

$= \frac{2}{k\pi} \sin(k\pi/2)$. So $a_{2n} = 0$, $a_{2n+1} = \frac{2(-1)^n}{(2n+1)\pi}$