LAST NAME:	FIRST NAME:

1 /12	2 /12	3 /10	4 /10	5 /10
6 /12	7 /8	8 /12	9 /14	T /100

MATH 4362 (Spring 2018), Final Exam, (Zweck)

Instructions: This 2 hour 45 minute exam is worth 100 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

Throughout this exam we define

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(1) [12 pts] **True or false? Give brief explanations for your answers.**

(a) Suppose that $u = u(t, x)$ solves

$$\begin{aligned} u_t + u_x &= 0 \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ u(0, x) &= \chi_{[-1,1]}(x) \qquad \text{for } x \in \mathbb{R}. \end{aligned}$$

Then the function $v(x) = u(1, x)$ is differentiable.

(b) Suppose that $u = u(t, x)$ solves

$$u_t = u_{xx} \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R},$$
$$u(0, x) = \chi_{[-1,1]}(x) \qquad \text{for } x \in \mathbb{R}.$$

Then the function $v(x) = u(1, x)$ is differentiable.

(c) Suppose that $u = u(t, x)$ solves

$$u_{tt} = u_{xx} \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R},$$
$$u(0, x) = \chi_{[-1,1]}(x) \qquad \text{for } x \in \mathbb{R},$$
$$u_t(0, x) = 0 \qquad \text{for } x \in \mathbb{R}.$$

Let $w(t) = u(t, 2)$. Then there is a time $T > 0$ so that $w(t) = 0$ for all $0 < t < T$.

(2) [12 pts] Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \frac{\pi}{2} - x$ for $x \in (0, \pi)$ and $f(\pi) = 0$. Let $\widetilde{f}$ be the 2π-periodic odd extension of f. Graph $\widetilde{f}$. Explain why the Fourier series of $\widetilde{f}$ converges pointwise but not uniformly on $\mathbb{R}$. What is the value of the Fourier series of $\widetilde{f}$ at (i) $x = 0$ and (ii) $x = \frac{5\pi}{4}$?

(3) [10 pts] Solve for $u = u(t, x)$ on $t \geq 0$ and $x \in \mathbb{R}$:

$$\begin{aligned} u_{tt} &= 4u_{xx} + \sin t, \\ u(0, x) &= e^{-x^2}, \\ u_t(0, x) &= 0. \end{aligned}$$

(4) [10 pts] Prove that for each $t > 0$ the series,

$$u(t,x) \;=\; \sum_{k=0}^{\infty} e^{-k^2 t} \cos(kx),$$

converges uniformly for $x \in \mathbb{R}$. Hence show that u is continuous where $t > 0$.
Hint: Fix $\epsilon > 0$. Prove that for all $t > \epsilon$ the series converges uniformly for $x \in \mathbb{R}$.

(5) [10 pts] Suppose that $u = u(t, x)$ solves

$$\begin{aligned} u_t &= u_{xx}, \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ u(0, x) &= \arctan(x). \end{aligned}$$

Let $v(t, x) = u_t(t, x)$. Show that v solves

$$\begin{aligned} v_t &= v_{xx}, \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ v(0, x) &= \frac{-2x}{(1 + x^2)^2}. \end{aligned}$$

(6) [12 pts] Suppose that $u = u(t, x)$ solves $u_t + (1 + x^2)u_x = 0$. Find and sketch the characteristic curves. Shade that portion of the (t, x)-plane with $t > 0$ where the solution is determined by the values of u at $t = 0$. Derive a formula for the solution, $u = u(t, x)$, with initial values $u(0, x) = f(x)$.

(7) [8 pts] Let $h(x,y) = \chi_{[-\pi/4,\pi/4]}(\theta)$ where $(x,y) = (\cos\theta, \sin\theta)$. Let $u = u(x,y)$ solve Laplace's equation

$$\begin{aligned} \triangle u = 0 \qquad & \text{in } x^2 + y^2 < 1, \\ u = h \qquad & \text{on } x^2 + y^2 = 1. \end{aligned}$$

True or false? Give brief explanations for your answers. In the following, $u = u(r,\theta)$.

(a) $u(0,0) < u(1,0)$ (b) $u(0,0) < u(1,\pi)$ (c) $u(0.9,\pi) < u(0.9,0)$.

Hint: The solution is given in polar coordinates by

$$u(r,\theta) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,K(r,\theta-\phi)\,d\phi, \qquad \text{where } K(r,\theta) \;=\; \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

(8) [12 pts] Let $C_0^\infty(\mathbb{R})$ be the space of infinitely differentiable functions, $u : \mathbb{R} \to \mathbb{R}$, with the property that there exists an $R > 0$ so that $u(x) = 0$ for all $|x| > R$.

(a) Define what it means for $L : C_0^\infty(\mathbb{R}) \to \mathbb{R}$ to be a distribution.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function and $u \in C_0^\infty(\mathbb{R})$. Show that $L_g(u) = \int_{-\infty}^{\infty} g(x)u(x)\,dx$ is a distribution.

(c) Let $\xi \in \mathbb{R}$. Define the Dirac delta distribution, δ_ξ, at $x = \xi$, and show that δ_ξ is indeed a distribution.

(d) Let σ_ξ be the piecewise continuous function defined by

$$\sigma_\xi(x) \;=\; \begin{cases} 0 & \text{if } x \leq \xi \\ 1 & \text{if } x > \xi. \end{cases}$$

Show that the derivative of the distribution L_{σ_ξ} equals the Dirac distribution, δ_ξ.

(9) [14 pts] Find a Fourier series solution, $u = u(t, x)$, for $t > 0$ and $x \in [0, \pi]$, of

$$\begin{aligned} u_t &= u_{xx}, \\ u_x(t, 0) &= 0, \\ u_x(t, \pi) &= 0, \\ u(0, x) &= \chi_{[0,\pi/2]}(x). \end{aligned}$$

You may assume the eigenvalues satisfy $\lambda \geq 0$.