LAST NAME:	FIRST NAME:

1 /12	2 /12	3 /12	4 /18	5 /12	6 /9	T /75

MATH 4362 (Spring 2018), Midterm Exam Two, (Zweck)

Instructions: This 75 minute exam is worth 75 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

Throughout this exam $\chi_{[a,b]}$ is the function defined by

$$\chi_{[a,b]}(x) \ = \ \begin{cases} 1 & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(1) [12 pts] Prove that $\{\phi_k(x) = e^{ikx} \ \big| \ k = 0, \pm 1, \pm 2, \cdots\}$ is an orthonormal set of functions on $[-\pi, \pi]$ with respect to the L^2-inner product.

(2) [12 pts] Find a formula for the solution $u = u(t, x)$ of the PDE initial-value problem

$$\begin{aligned} u_{tt} &= \tfrac{1}{4} u_{xx} \\ u(0, x) &= \chi_{[-1,1]}(x) \\ u_t(0, x) &= 0. \end{aligned}$$

Sketch the solution at $t = 1$ and at $t = 4$.

(3) [12 pts] Find a formula for the solution $u = u(t, x)$ of the PDE initial-value problem

$$\begin{aligned} u_{tt} &= u_{xx} \\ u(0, x) &= 0 \\ u_t(0, x) &= \cos(x). \end{aligned}$$

Sketch the solution at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

(4) [18 pts] Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by $f(x) = \chi_{[-\pi/2,\pi/2]}(x)$.

(a) Calculate the Fourier series of f.

(b) Apply the theorem on pointwise convergence of Fourier series to show that the Fourier series you derived in (a) converges to a function $F : \mathbb{R} \to \mathbb{R}$. Sketch the graph of F on the domain $[-2\pi, 2\pi]$.

(c) With the aid of a (rough) sketch, discuss what the Gibb's phenomenon has to say about how well the partial sums of the Fourier series of f approximate the function F near $x = \frac{\pi}{2}$.

(5) [12 pts] The function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = x^2$ has Fourier series

$$x^2 \sim \frac{2\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx). \tag{1}$$

(a) Apply the theorem on the differentiation of Fourier Series to show that the Fourier series (1) can be differentiated term-by-term.

(b) What happens when you differentiate the Fourier series (1) term-by-term a second time? In particular, does the theorem on the differentiation apply to the Fourier series of the function $g(x) = f'(x) = 2x$?

(6) [9 pts] Suppose that $u = u(t, x)$ satisfies the inhomogeneous wave equation

$$\begin{aligned} u_{tt} - 9u_{xx} &= F(t, x) \\ u(0, x) &= 0 \\ u_t(0, x) &= 0, \end{aligned}$$

where $F(t, x) = \chi_{[0,1]}(x)$ for all $t > 0$. Use the concept of the domain of dependence to show that $u(2, 8) = 0$. Then find all x so that $u(2, x) = 0$.