LAST NAME: SOLUTIONS	FIRST NAME:

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MATH 4362 (Spring 2018)

Midterm Exam One

(Zweck)

Instructions: This 75 hour exam is worth 75 points. No books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

(1) [15 pts] Complete the table.

PDE	Order	Equilibrium or Dynamic	Linear or Nonlinear	Homogeneous or Inhomogeneous	Name
$u_t - 4u_{xx} = 0$	2	D	L	H	HEAT EQN
$u_{tt} - 16u_{xx} = \sin x \cos t$	2	D	L	I	WAVE EQN
$u_t + uu_x = 0$	1	D	NL	H	INVISCID BURGERS OR NONLINEAR TRANSPORT
$u_{xx} + u_{yy} = f(x, y)$	2	E	L	I	POISSON EQN
$u_t + e^{-x}u_x + u = 0$	1	D	L	H	TRANSPORT OR 1-WAY WAVE EQN WITH DECAY

(2) [15 pts] Solve the initial value problem for $u = u(t, x)$ given by

$$u_t - 4u_x + 3u = 0,$$
$$u(0, x) = e^{-x^2}.$$

LET $\dfrac{dx}{dt} = -4 \Rightarrow x = -4t + \xi \Rightarrow x + 4t = \xi$

LET $h(t) = u(t, x(t)) = u(t, -4t + \xi)$

Then

$h'(t) = u_t - 4u_x = -3u = -3h(t)$

ALSO $h(0) = u(0, \xi) = e^{-\xi^2}$.

So we have ODE IVP for h to solve.

$\displaystyle\int \frac{dh}{h} = -3\int dt$

$\ln|h| = -3t + c$

$|h| = e^{-3t+c}$

$h(t) = Ae^{-3t} \quad A = \pm e^c \in \mathbb{R}$

$e^{-\xi^2} = h(0) = A$

So $u(t, -4t+\xi) = h(t) = e^{-\xi^2} e^{-3t} \qquad \xi = x + 4t$

$$\boxed{u(t, x) = e^{-(x+4t)^2} e^{-3t}}$$

(3) [15 pts] Consider the PDE for $u = u(t, x)$ given by $u_t + x^2 u_x = 0$.
(a) By solving the ODE for the characteristics show that the characteristic curve that goes through the point (t_1, x_1) is given by

$$x = x(t) = \frac{x_1}{1 + x_1(t_1 - t)}. \quad (1)$$

$$\boxed{\begin{cases} \dfrac{dx}{dt} = x^2 \\ x(t_1) = x_1 \end{cases}}$$

① CASE $x_1 = 0$: $x(t) = 0$ is equilibrium solution ✓

② CASE $x_1 \neq 0$:

$$\int \frac{dx}{x^2} = \int dt$$

$$-x^{-1} = t + k$$

$$k = -\frac{1}{x} - t.$$

plug in $(t, x) = (t_1, x_1)$ to get

$$k = -\frac{1}{x_1} - t_1$$

So $x^{-1} = -t - k = -t + \frac{1}{x_1} + t_1$

OR

$$x(t) = \frac{1}{\frac{1}{x_1} + t_1 - t} = \frac{x_1}{1 + x_1(t_1 - t)} \checkmark$$

(b) Show that if $x_1 > 0$ then the characteristic curve in (1) intersects the x-axis. Sketch this curve when $(t_1, x_1) = (2, 1)$.

IF $x_1 > 0$ and $0 \leq t \leq t_1$ Then $1 + x_1(t_1 - t) \geq 1$

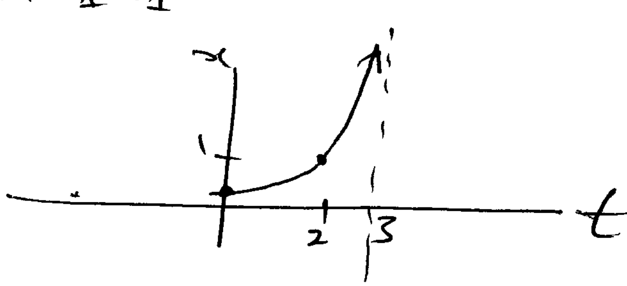
So $x(t) = \dfrac{x_1}{1 + x_1(t_1 - t)} \in \mathbb{R}$. (Denominator $\neq 0$)

So CC is CTS on $0 \leq t \leq t_1$.

ALSO $x(0) = \dfrac{x_1}{1 + x_1 t_1} \in \mathbb{R}$ as $1 + x_1 t_1 \geq 1$ for $x_1 > 0$, $t_1 > 0$

EX

$$x(t) = \frac{1}{3 - t}$$



(c) Suppose that $u(0, y) = \cos(y)$. Find a formula for the solution, $u = u(t, x)$, for $x \geq 0$ and $t \geq 0$.

Since u is constant along CC $u(t_1, x_1) = u(0, x(0))$

where $x(0) = \frac{x_1}{1+x_1 t_1}$ $= \cos(x(0))$

So $u(t_1, x_1) = \cos\left(\frac{x_1}{1+x_1 t_1}\right)$

or $u(t, x) = \cos\left(\frac{x}{1+xt}\right)$

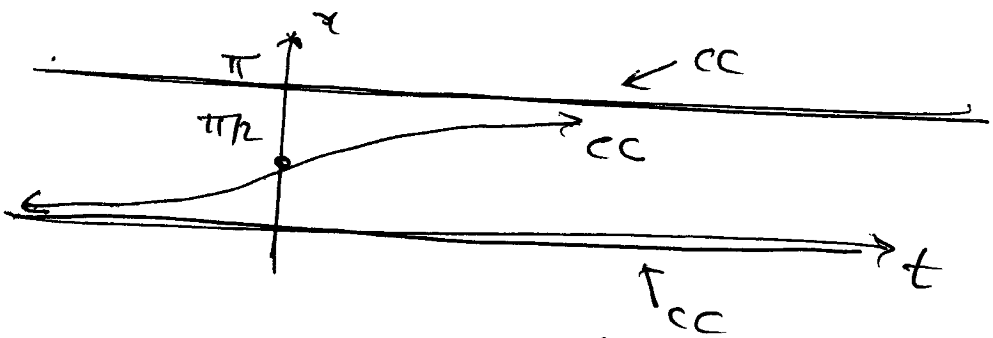
(4) [10 pts] Consider the PDE for $u = u(t, x)$ given by $u_t + (\sin x)u_x = 0$.
(a) Show that the horizontal lines $x = x(t) = 0$ and $x = x(t) = \pm\pi$ are characteristic curves.

$\frac{dx}{dt} = \sin x$

$\boxed{x=0}$ $\frac{dx}{dt} = 0$, $\sin x = 0$ So $x = 0$ is a CC

$\boxed{x = \pm\pi}$ $\frac{dx}{dt} = 0$, $\sin x = 0$, So $x = \pm\pi$ is a CC

(b) Show that the characteristic, $x = x(t)$, passing through $(t, x) = (0, \pi/2)$ is an increasing function of t.



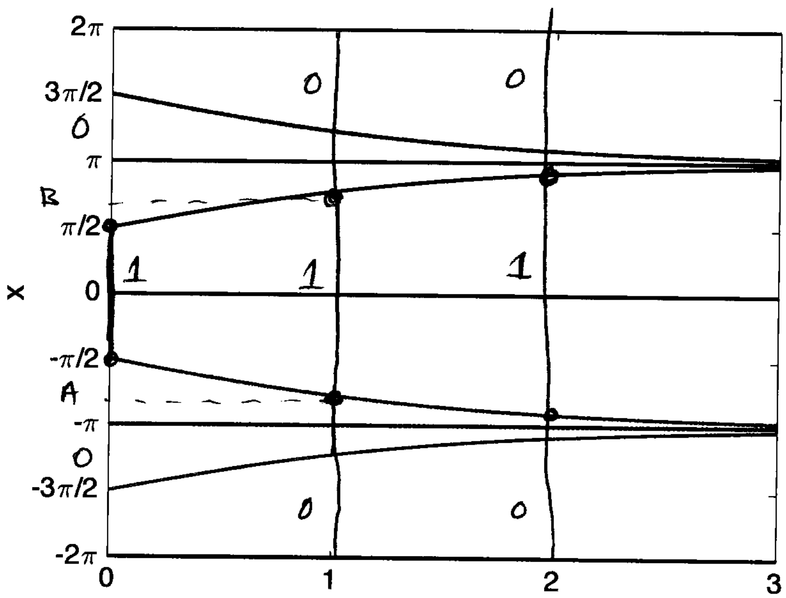
Since CCs cannot cross (being soln of a ODE IVP)
the CC thru $(0, \pi/2)$ must lie between lines $x = 0, x = \pi$
(which are both CCs by (a)).
In this region $0 < x < \pi$ So $\sin x > 0$
So $\frac{dx}{dt} = \sin x > 0$ So $x = x(t)$ is ↑.

(c) Suppose now that $u = u(t, x)$ solves the initial value problem

$$u_t + (\sin x)u_x = 0,$$

$$u(0, x) = \begin{cases} 1 & \text{if } |x| \le \frac{\pi}{2}, \\ 0 & \text{if } |x| > \frac{\pi}{2}. \end{cases}$$

Use the sketch of the characteristic curves below to sketch the solution u at times $t = 1$ and $t = 2$. What is $u_\infty(x) = \lim_{t\to\infty} u(t, x)$?

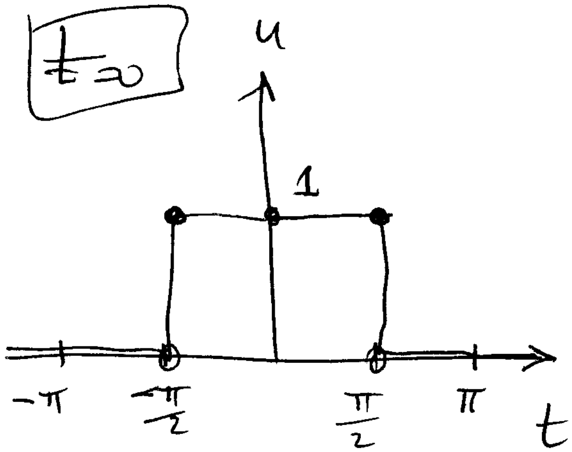


USE FACT

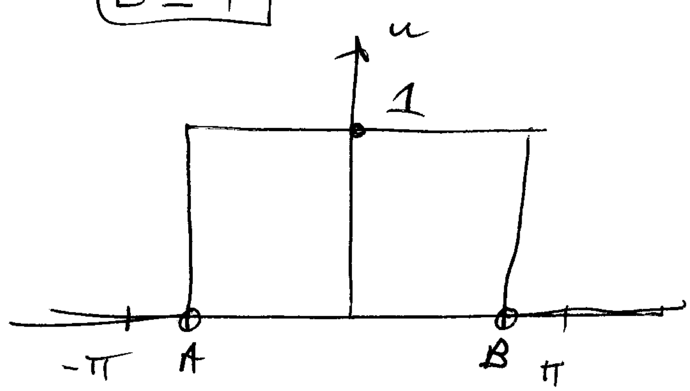
u IS CONST

ALONG CCs

$\boxed{t=0}$



$\boxed{t=1}$

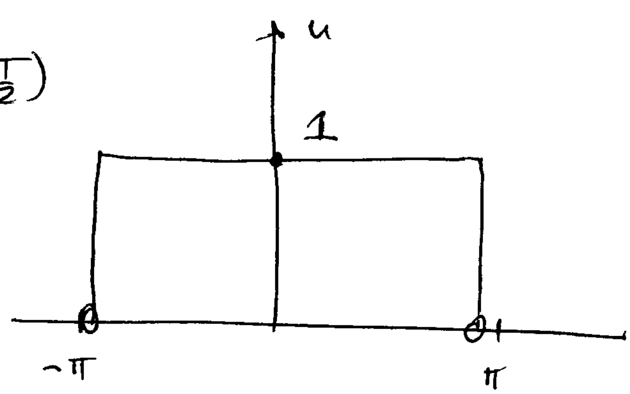


As $t \to \infty$, the CC thru $(0, \pm\frac{\pi}{2})$

does $x \to \pm\pi$.

So

$$u_\infty = \begin{cases} 1 & |x| < \pi \\ 0 & |x| \ge \pi \end{cases}$$



NOTE $u(t, \pm\pi) = 0 \ \forall t$ by (a)

(5) $u_t + 3uu_x = 0$, $x \in \mathbb{R},\ t \geq 0$

$$u(0,x) = \begin{cases} -2 & \text{IF } x < 1 \\ 0 & \text{IF } x \geq 1. \end{cases}$$

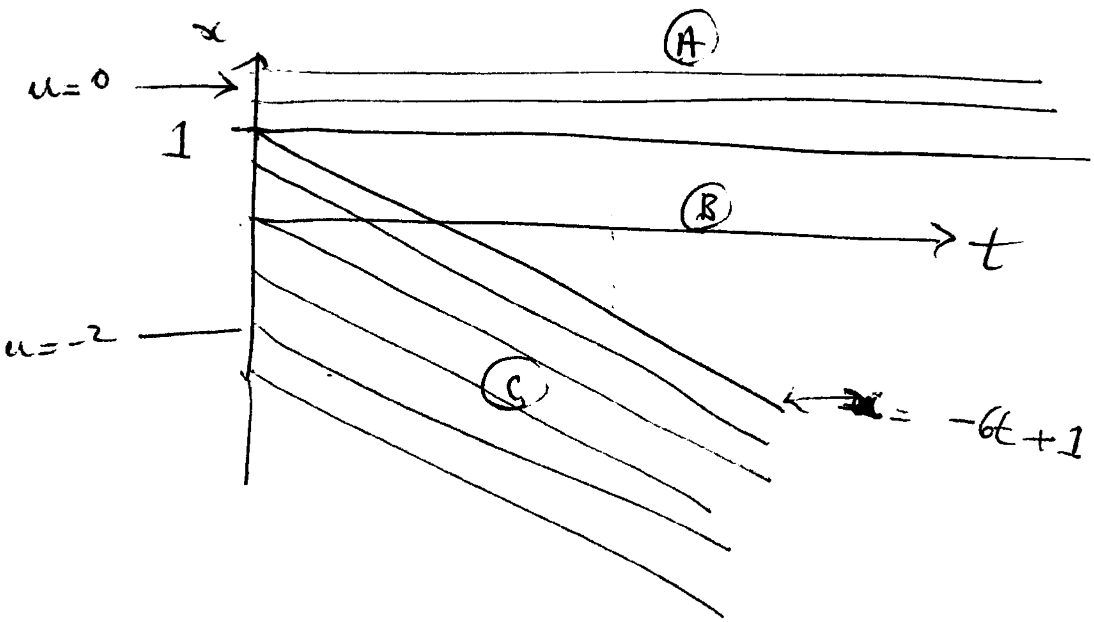
CCs are $x = 3ut + \xi$ and u is CONST on each CC.

FOR $\xi < 1$, CC Thru $(0,\xi)$ has $u = -2$

So eqn is $x = -6t + \xi$

FOR $\xi \geq 1$ CC Thru $(0,\xi)$ has $u = 0$

So eqn is $x = \xi$



(A) : $u = 0$

(B) Cannot determine u

(C) $u = -2$

$$u(t,x) = \begin{cases} 0 & x \geq 1,\ t \geq 0 \\ \text{NOT DETERMINED} & \cancel{x < 1},\ -6t+1 \leq x < 1,\ t > 0 \\ -2 & x < -6t+1,\ t > 0 \end{cases}$$

(6) [10 pts] Suppose that $u = u(t, x)$ is a solution of the PDE

$$u_{tt} - c^2 u_{xx} = 0. \tag{1}$$

Let $\xi = x - ct$ and $\eta = x + ct$, and define a function $v = v(\xi, \eta)$ by

$$v(\xi, \eta) = u\left(\frac{\eta - \xi}{2c}, \frac{\eta + \xi}{2}\right), \text{ so } u(t,x) = v(x-ct, x+ct)$$

Prove that u solves (1) if and only if $v_{\xi\eta} = 0$. Hence show that any solution of (1) is of the form,

$$u(t, x) = p(x - ct) + q(x + ct),$$

for some functions p and q.

$$\xi = x - ct, \quad \eta = x + ct$$

$$\frac{\eta - \xi}{2c} = t, \quad \frac{\eta + \xi}{2} = x$$

So

$$u(t,x) = v(x-ct, x+ct)$$

$$u_t = -cv_\xi + cv_\eta$$

$$u_{tt} = (-c)^2 v_{\xi\xi} + c^2 v_{\eta\eta} - c^2 v_{\xi\eta} + -c^2 v_{\eta\xi}$$

$$u_{tt} = c^2[v_{\xi\xi} + v_{\eta\eta} - 2v_{\xi\eta}]$$

$$u_{xx} = v_{\xi\xi} + v_{\eta\eta} + 2v_{\xi\eta}$$

$$0 = u_{tt} - c^2 u_{xx} = 4c^2 v_{\xi\eta}. \checkmark$$

Let $w = v_\eta$.

$$w_\xi = 0 \Rightarrow w(\xi, \eta) = \tilde{q}(\eta)$$

$$\frac{\partial v}{\partial \eta} = \tilde{q}(\eta) \Rightarrow v(\xi, \eta) = \int \tilde{q}(\eta)\, d\eta + p(\xi) = q(\eta) + p(\xi)$$

$$\text{So } u(t,x) = p(x-ct) + q(x+ct)$$