LECTURE 9 **CONVERGENCE OF FOURIER SERIES**

COMPLEX FOURIER SERIES

EULER'S FORMULA

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

$$e^{-ikx} = \cos(kx) - i\sin(kx)$$

So

$$\cos(kx) = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$\sin(kx) = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

LET'S PLAY

IDEA: CALCULATIONS OFTEN EASIER WITH COMPLEX EXPS THAN WITH TRIG FNS

$$a_k \cos kx + b_k \sin kx$$

$$= \frac{1}{2}a_k\left(e^{ikx} + e^{-ikx}\right) + \frac{1}{2i}b_k\left(e^{ikx} - e^{-ikx}\right)$$

$$= \frac{1}{2}(a_k - ib_k)e^{ikx} + \frac{1}{2}(a_k + ib_k)e^{-ikx}$$

So set

$$c_k = \frac{1}{2}(a_k - ib_k) \qquad k > 0$$

$$c_{-k} = \frac{1}{2}(a_k + ib_k) \qquad k > 0$$

$$c_0 = \frac{1}{2}a_0$$

Then

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$= c_0 e^{i0x} + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k} e^{-ikx}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

and

$$c_k = \frac{1}{2}(a_k - i b_k)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\cos kx - i \sin kx] \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{e^{ikx}} \, dx$$

$$c_k = \langle f, e^{ikx} \rangle$$

SUMMARY

1. Let $f, g : [-\pi, \pi] \to \mathbb{C}$. Define HERMITIAN L^2-INNER PRODUCT by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx.$$

(3) NORM of f: $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx}$.

(3) $\{e^{ikx} \mid k \in \mathbb{Z}\}$ is ONB in that

$$\langle e^{ikx}, e^{ilx}\rangle = \delta_{kl}.$$

PF

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\overline{e^{ilx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)x}\,dx$$

$$= \begin{cases} \frac{1}{2\pi}\left[\frac{e^{i(k-l)x}}{i(k-l)}\right]_{-\pi}^{\pi} = 0 & \text{IF } k \neq l \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = 1 & \text{IF } k = l. \end{cases}$$

(4) Let $f : [-\pi, \pi] \to \mathbb{C}$. The COMPLEX FS of f is

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

where

$$c_k = \langle f, e^{ikx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

EX $f(x) = e^{ax}$, $a \neq 0$.

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-ikx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-ik)x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-ik)x}}{a-ik}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\cancel{2\pi}\,2}\;\frac{e^{(a-ik)\pi} - e^{-(a-ik)\pi}}{(a-ik)\pi} \qquad e^{ik\pi} = (-1)^k$$

$$= (-1)^k \frac{e^{a\pi} - e^{-a\pi}}{2\pi\,(a-ik)}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$= (-1)^k \frac{\sinh(a\pi)}{\pi\,(a-ik)}$$

$$= \frac{(-1)^k (a+ik)\sinh(a\pi)}{\pi\,(a^2+k^2)}$$

So
$$e^{ax} \sim \frac{\sinh(a\pi)}{\pi}\sum_{k=-\infty}^{\infty} \frac{(-1)^k (a+ik)}{a^2+k^2}\, e^{ikx} \qquad (*)$$

NOTICE If set $a=1$, $x=0$ get

$$\frac{\pi}{\sinh(\pi)} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k(1+ik)}{1+k^2}$$

$$= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1+k^2} + i \sum_{k=-\infty}^{\infty} \frac{(-1)^k k}{1+k^2}$$

$$= 1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2}$$

as $\displaystyle\sum_{k=-\infty}^{-1} \frac{(-1)^k k}{1+k^2} \underset{l=-k}{=} -\sum_{l=1}^{\infty} \frac{(-1)^l l}{1+l^2}$.

So $\displaystyle\sum_{k=-\infty}^{\infty} \frac{(-1)^k k}{1+k^2} = 0$ HOLDS!

EXERCISE Convert $\circledast$ to F.S. involving TRIG f^{ns}.

POINTWISE + UNIFORM CONVERGENCE

FOR PROOFS SEE AN ANALYSIS COURSE LIKE MATH 4301/4302

DEF ① Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of f^{ns}.
We say $f_n \to f$ POINTWISE if

$$\forall x \in [a,b] \quad f_n(x) \to f(x) \quad \text{as } n \to \infty$$

ie if $\forall x \in [a,b], \quad \forall \varepsilon > 0, \quad \exists N : \forall n \geq N$

$$|f_n(x) - f(x)| < \varepsilon. \qquad N = N(\varepsilon, x)$$

② Let $f_n : [a,b] \to \mathbb{R}$. We say $f_n \to f$ UNIFORMLY on $[a,b]$ if

$$\forall \varepsilon > 0, \quad \exists N : \forall x \in [a,b], \quad \forall n \geq N$$

$$|f_n(x) - f(x)| < \varepsilon \qquad N = N(\varepsilon)$$

EX

$$f_n(x) = x^n \quad \text{on } [0,1]$$

$$f(x) = \begin{cases} 0 & \text{IF } x < 1 \\ 1 & \text{IF } x = 1 \end{cases}$$

Then

$f_n \to f$ POINTWISE but NOT UNIFORMLY.

as for $x < 1$

$$|f_n(x) - f(x)| = |x^n - 0| = |x|^n < \varepsilon$$

$$\iff n \log|x| < \log \varepsilon \quad (\log \text{ is } \uparrow)$$

$$\iff n > \frac{\log \varepsilon}{\log|x|} \quad \text{as } \log|x| < 0 \text{ for } 0 < x < 1$$

Set $N(\varepsilon, x) = \dfrac{\log \varepsilon}{\log|x|}$

As $x \to 1$, $N(\varepsilon, x) \to \infty$.

So don't expect uniform convergce.

$\boxed{\text{THM A}}$ If $f_n \to f$ UNIF on $[a,b]$ and each f_n is CTS on $[a,b]$ Then f is CTS on $[a,b]$.

__NOTE__ EX above shows PW convergce not enough to ensure limit is CTS.

DEF Let $u_k : [a,b] \longrightarrow \mathbb{R}$, $k = 1, 2, \ldots$

and $S_n(x) = \sum_{k=1}^{n} u_k(x)$.

We say $\sum_{k=1}^{\infty} u_k$ CONVERGES (PW/UNIF) if

$S_n \longrightarrow S$ PW / UNIF

WEIERSTRASS M_TEST

Suppose $|u_k(x)| \leq m_k \quad \forall x \in [a,b]$.
[Each u_k is bounded]

If $\sum_{k=1}^{\infty} m_k$ CONVS.

Then $\sum_{k=1}^{\infty} u_k(x) = f(x)$

CONVS UNIFLY + ABSOLUTELY.

Consequently (by Thm A) if each u_k is CTS, so is f.

We can manipulate UNIFLY CONVGT series just like finite sums.

THM B

Suppose $f(x) = \sum_{k=1}^{\infty} u_k(x)$ CONVGS UNIF

Then

1. $\oint g(x) f(x) = \sum_{k=1}^{\infty} g(x) u_k(x)$

   for any bounded f^n g.

2. $$\int_a^b f(y)\,dy = \int_a^b \left[\sum_{k=1}^{\infty} u_k(y)\right] dy$$

   $$= \sum_{k=1}^{\infty} \int_a^b u_k(y)\,dy$$

   – Can integrate series term by term

THM C

Suppose $f(x) = \sum_{k=1}^{\infty} u_k(x)$ POINTWISE

and $\sum_{k=1}^{\infty} u_k'(x) =: g(x)$ CONVS UNIF

Then $\sum_{k=1}^{\infty} u_k(x)$ CONVS UNIF and

$$f'(x) = g(x)$$

i.e.

$$\frac{d}{dx}\left[\sum_{k=1}^{\infty} u_k(x)\right] = \sum_{k=1}^{\infty} \frac{du_k}{dx}(x)$$

– TERM BY TERM DIFFN.

$\boxed{\text{THM D}}$

Suppose $f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$

IF $\sum_{k=1}^{\infty} |c_k|^2 < \infty$

Then the F.S. CONVS UNIF to a CTS $\tilde{f}$

WITH same FS coefficients as f:

$$c_k = \langle \tilde{f}, e^{ikx} \rangle = \langle f, e^{ikx} \rangle$$

PF

$|c_k e^{ikx}| \leq c_k$ as $|e^{ikx}| = 1$

So $\hat{f}(x) = \sum c_k e^{ikx}$ CONVS by Weierstrass Test.

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\,e^{-ilx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\sum_k c_k e^{ikx}\Big]e^{-ilx}\,dx$$

$$\overset{\text{THM B1}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\sum_k c_k e^{i(k-l)x}\Big]dx$$

$$\overset{\text{THM B2}}{=} \frac{1}{2\pi}\sum_k \int_{-\pi}^{\pi} c_k e^{i(k-l)x}\,dx$$

$$= \sum_k c_k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-l)x}\,dx$$

$$= \sum_k c_k \delta_{kl} = c_l.$$

$\square$

$\boxed{\text{THM E}}$

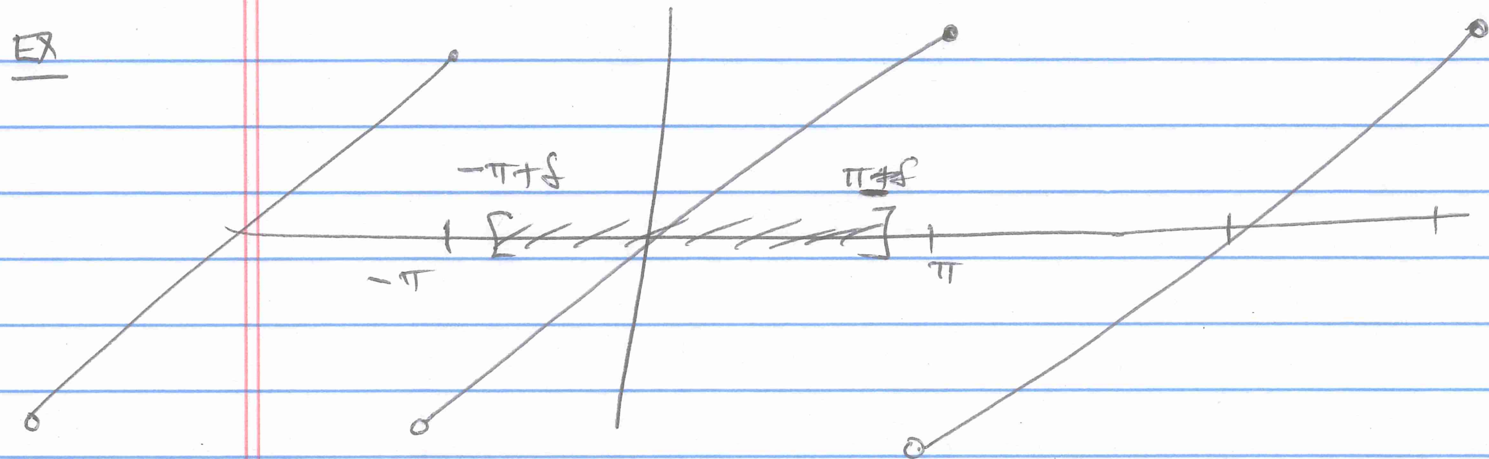
Let f be 2π-periodic and piecewise C^1.

If f is CTS on subinterval (a,b)

Then F.S. of f CONVS UNIF to f

on any closed subinterval $[a+\delta, b-\delta]$ of (a,b).

EX



F.S. converges UNIF to $f(x) = x$ on $[-\pi+\delta, \pi-\delta]$

– Related to Gibbs Phenomenon only a problem at discties $x = \pm\pi$.

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$\boxed{\text{THM F}}$

Suppose $$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Then

① $$\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \|f\|^2$$ BESSEL'S INEQUALITY

② Hence $c_k \to 0$ as $k \to \pm\infty$.

Also $a_k \to 0$, $b_k \to 0$ as $k \to \infty$
as $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$

PF Let $\varphi_k(x) = e^{ikx}$ and

$$s_n = \sum_{|k| \le n} c_k \varphi_k$$

NOW

A $\|s_n\|^2 = \langle s_n, s_n \rangle$

$$= \left\langle \sum_{|k| \le n} c_k \varphi_k , \sum_{|l| \le n} c_l \varphi_l \right\rangle$$

$$= \sum_{|k|, |l| \le n} c_k \overline{c_l} \langle \varphi_k, \varphi_l \rangle$$

$$= \sum_{|k|, |l| \le n} c_k \overline{c_l} \, \delta_{kl} = \sum_{|k| \le n} |c_k|^2$$

B $\langle f, s_n \rangle = \left\langle f, \sum_{|k| \le n} c_k \varphi_k \right\rangle$

$$= \sum_{|k| \le n} \overline{c_k} \langle f, \varphi_k \rangle$$

$$= \sum_{|k| \le n} |c_k|^2 = \|s_n\|^2.$$

③ $0 \le \|f - s_n\|^2$

$$= \langle f - s_n, f - s_n \rangle$$

$$= \|f\|^2 + \|s_n\|^2 - 2\langle f, s_n \rangle$$

$$\overset{①,②}{=} \|f\|^2 - \sum_{|k| \le n} |c_k|^2 \quad ✓$$

So $\displaystyle\sum_{|k| \le n} |c_k|^2 \le \|f\|^2$.

④ $a_n = \displaystyle\sum_{|k| \le n} |c_k|^2$ is $\uparrow$ seq, bounded above. So $a_n \to a_\infty = \displaystyle\sum_{k=-\infty}^{\infty} |c_k|^2$ convs.

So $c_k \to 0$ must hold as $k \to \pm\infty$.

$\square$