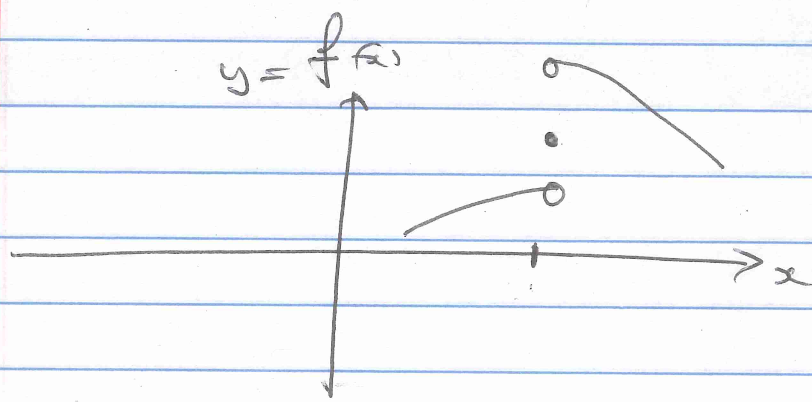
LECTURE 8 FOURIER SERIES, PART II

THM ON POINTWISE CONVERGENCE OF F.S

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic, piecewise C^1 function.

Then at any $x \in \mathbb{R}$, the Fourier Series of f converges to

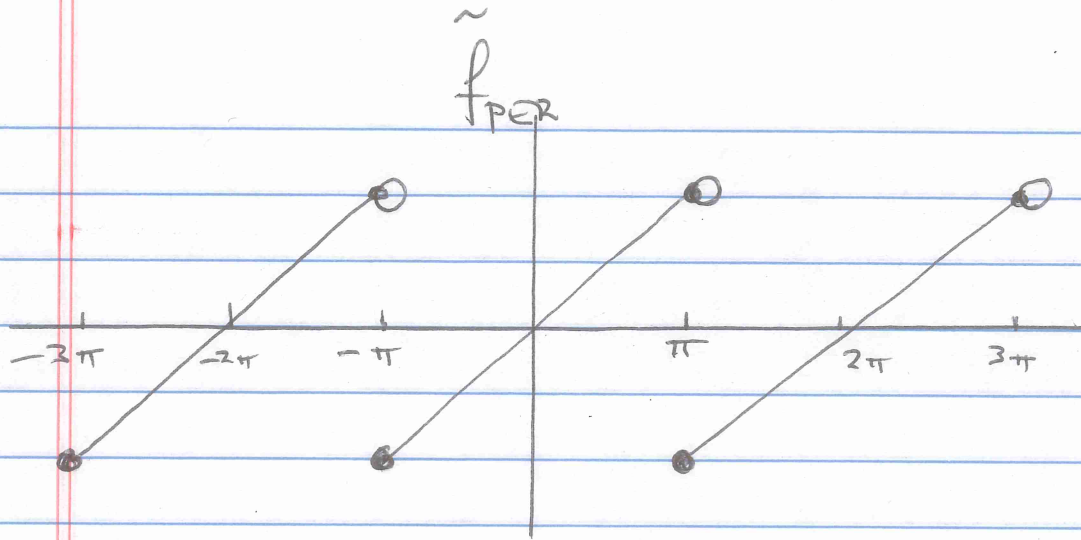
$$\begin{cases} f(x) & \text{if } f \text{ is CTS at } x \\ \frac{1}{2}\left[f(x+) + f(x-)\right] & \text{if } f \text{ has JUMP DISCTY at } x \end{cases}$$



EX LET $f : [-\pi, \pi] \to \mathbb{R}$ be defined by

$$f(x) = x \quad \text{for } -\pi \leq x \leq \pi$$

Extend f to a 2π-periodic PW C^1 f^n, which we will ~~deno~~ call f_{PER}



$$\tilde{f}_{PER}(x) = x - 2m\pi \quad \text{for } (2m-1)\pi \leq x < (2m+1)\pi$$
where $m \in \mathbb{Z}$.

CHECK
- Slope $= 1$
- Crosses x axis at $x = 2m\pi$.

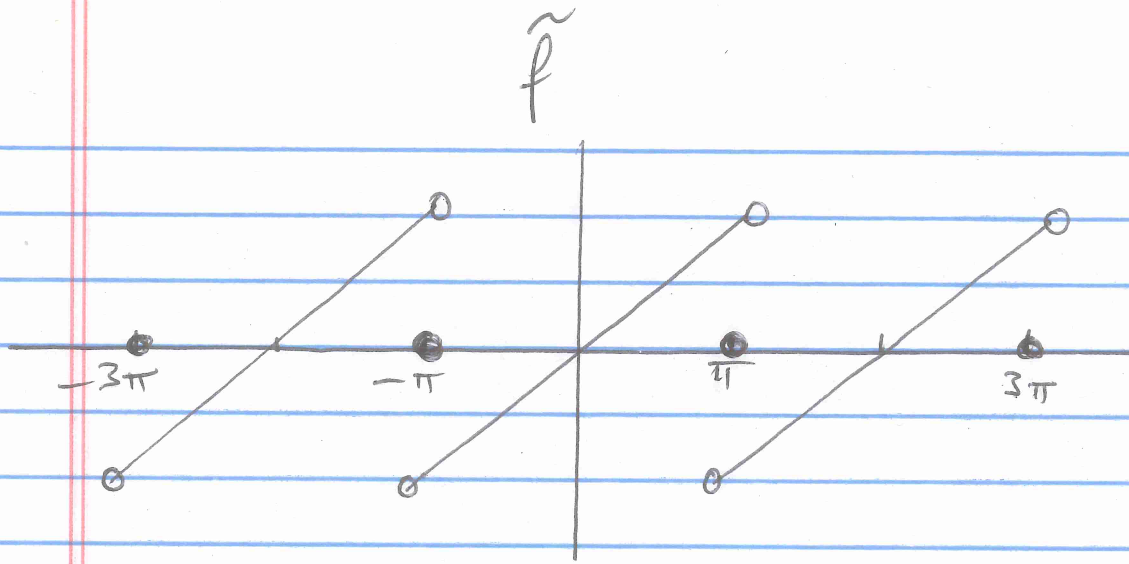
LAST TIME: Fourier Series of f is

$$x \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

SO PW CONVERGE THM SAYS

$$2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx) = \begin{cases} x - 2m\pi & \text{IF } (2m-1)\pi < x < (2m+1)\pi \text{ FOR SOME } m \\ 0 & \text{IF } x = (2n-1)\pi \text{ for some } n \end{cases}$$

$$=: \tilde{f}(x)$$

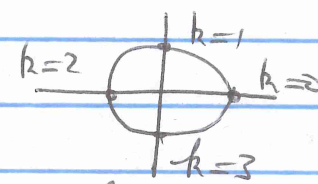


NOTICE

① $\tilde{f}(0) = 0$ ✓ $\hat{f}(\pi) = 0$.

② When $x = \pi/2$ we have

$$\frac{\pi}{2} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin\left(\frac{k\pi}{2}\right)$$



$k = 2m+1$

So

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^{2m+2}}{2m+1} (-1)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Although the FS converges at each x, The convergence is not UNIFORM:

We have $\hat{f}(x) = \lim_{n \to \infty} S_n(x)$ where

S_n = nth partial sum of FS.

BUT the # of terms, n, need to ensure error is less than ε,

$$|\hat{f}(x) - s_n(x)| < \varepsilon$$

depends on x:

CONVGE is only PW

$$\forall x \ \forall \varepsilon > 0 \quad \exists N = N(\varepsilon, x) : \quad \forall n \geq N \quad |s_n(x) - \hat{f}(x)| < \varepsilon$$

Rather than UNIFORM

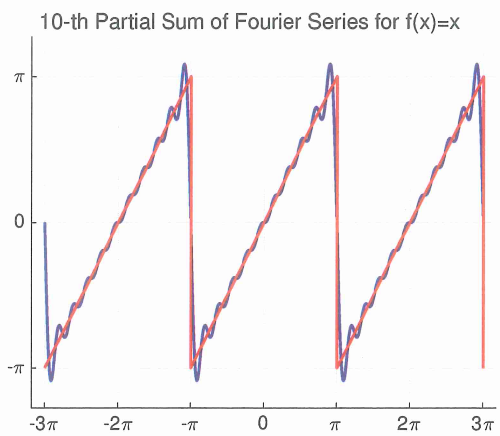
$$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : \quad \forall x, \ \forall n \geq N \ |s_n(x) - \hat{f}(x)| < \varepsilon.$$

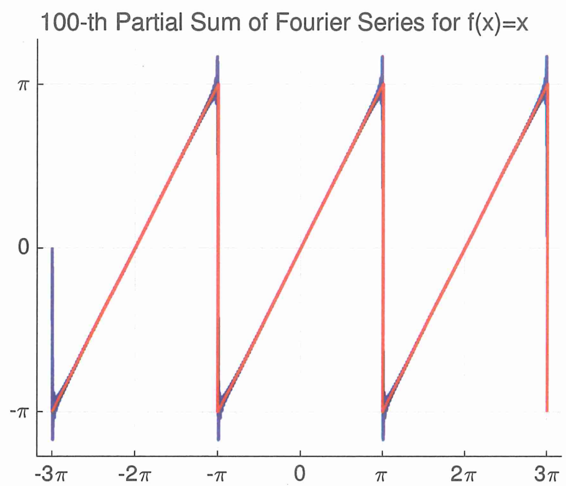
GIBB'S PHENOMENON

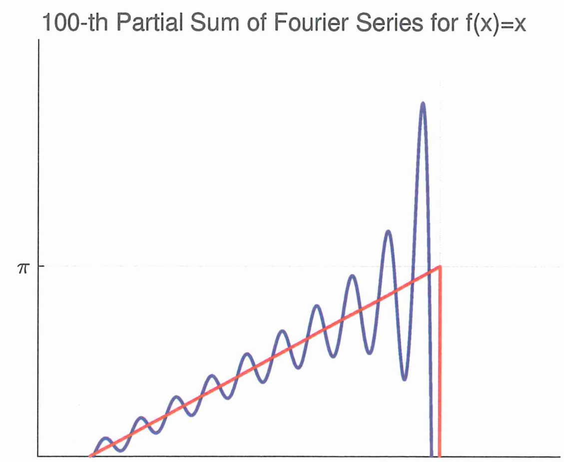
- Away from jump discontinuities, series converges.
- Near jumps $\exists$ overshoot of $\sim 9\%$ of jump magnitude
- As $n \uparrow$ ~~W~~width of overshoot region $\downarrow$, but magnitude persists, indept of n.

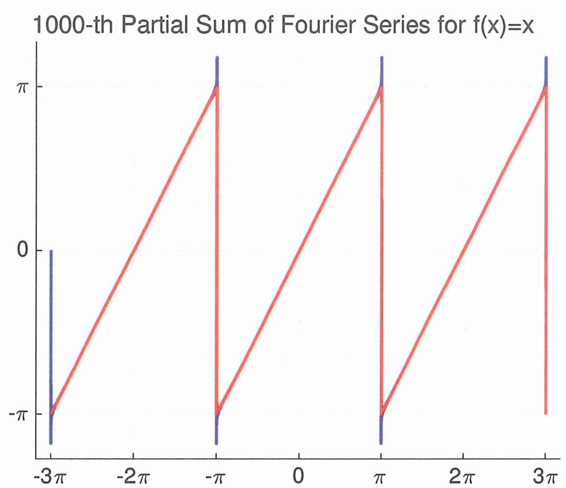
5

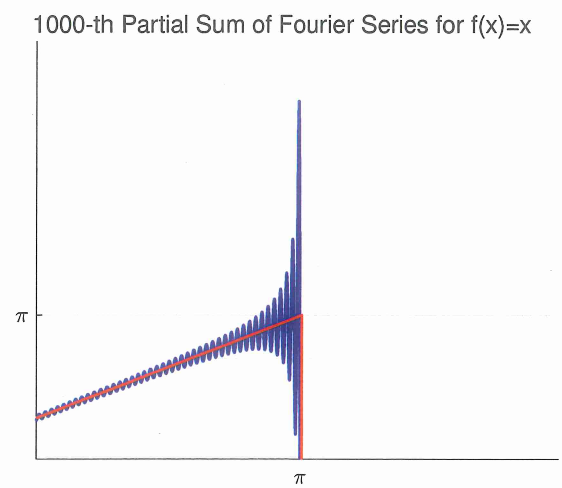
Fourier Series of Sawtooth Function and Gibbs Phenomenon











FOURIER COSINE SERIES

Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be even.

Then $b_k = \int_{-\pi}^{\pi} \underbrace{f(x) \sin(kx)}_{\text{EVEN} \times \text{ODD}} \, dx = 0 \quad \forall k$

So

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$ FOURIER COSINE SERIES

WITH $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx) \, dx$

FOURIER SINE SERIES

Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ be ODD

Then $a_k = \int_{-\pi}^{\pi} \underbrace{f(x) \cos(kx)}_{\text{ODD} \times \text{EVEN}} \, dx = 0 \quad \forall k$

So

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx)$$ FOURIER SINE SERIES

WITH $b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) \, dx$

CONSTRUCTION

1. Let $f : [0, \pi] \longrightarrow \mathbb{R}$. Then f has a UNIQUE EVEN extension to $\tilde{f}_{\text{EVEN}} : [-\pi, \pi] \longrightarrow \mathbb{R}$ defined by setting

$$\tilde{f}_{\text{EVEN}}(x) = f(x) \quad \text{for } 0 \le x \le \pi$$
$$\tilde{f}_{\text{EVEN}}(x) = f(-x) \quad \text{for } -\pi \le x < 0.$$

We can further extend $\tilde{f}_{EVEN}$ to a 2π-PERIODIC EVEN function

$$\hat{\tilde{f}}_{EVEN,\,PER} : \mathbb{R} \to \mathbb{R}.$$

<u>FACT</u>

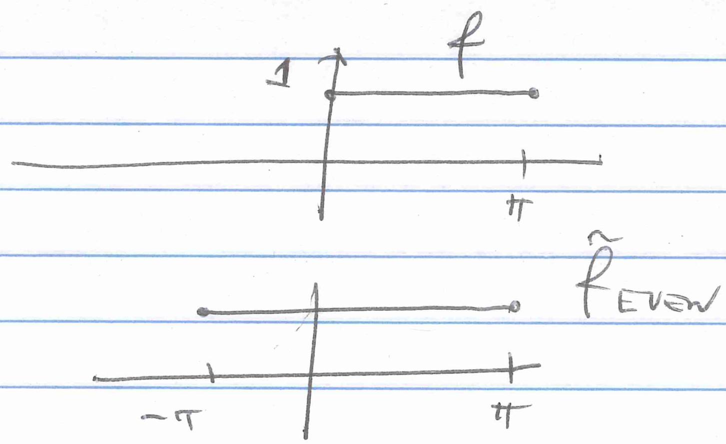
$$\tilde{f}_{EVEN,\,PER}(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

with

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)\, dx.$$

The Fourier Cosine Series of f represents the even 2π-periodic extension of f.

<u>EX</u>



So $\tilde{f}_{EVEN,\,PER}(x) = 1 \quad \forall x \in \mathbb{R}.$

We have

$$a_k = \frac{2}{\pi} \int_0^{\pi} 1 \cos(kx)\, dx = \begin{cases} 2 & k=0 \\ 0 & k>0 \end{cases}$$

So Fourier Cosine Series of $f = \frac{a_0}{2} = 1$ ✓

(2) Let $f : [0, \pi] \to \mathbb{R}$ WITH $f(0) = 0$. Then f has a UNIQUE ODD extension

$$\tilde{f}_{ODD} : [-\pi, \pi] \to \mathbb{R}$$

defined by

$$\tilde{f}_{ODD}(x) = f(x), \quad 0 \leq x \leq \pi$$

$$\tilde{f}_{ODD}(x) = -f(-x), \quad -\pi \leq x < 0.$$

We can further extend $\tilde{f}_{ODD}$ to a 2π-periodic ODD f^n

$$\tilde{f}_{ODD, PER} : \mathbb{R} \to \mathbb{R}.$$

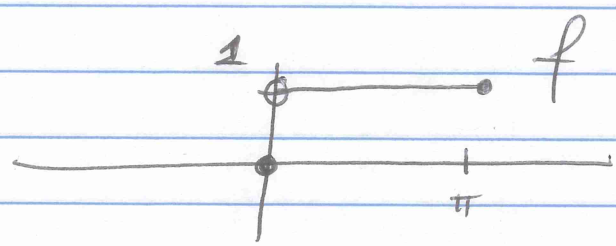
<u>FACT</u>

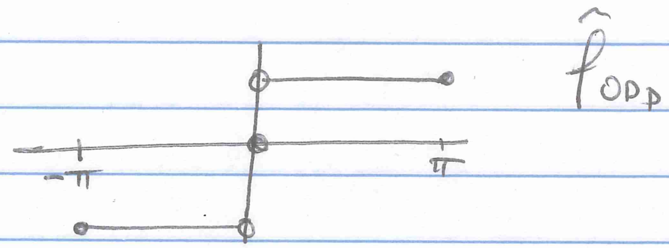
$$\tilde{f}_{ODD, PER}(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx)$$

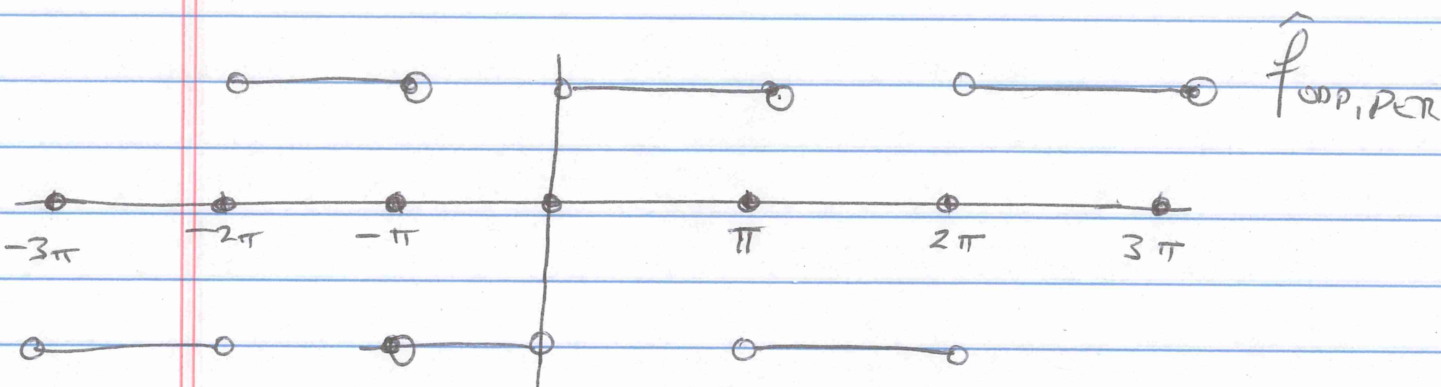
$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx$$

The Fourier Sine Series represents the odd 2π-periodic extension of f.

<u>EX</u>







$$b_k = \frac{2}{\pi}\int_0^{\pi} 1 \sin(kx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{-1}{k}\cos(kx)\right]_0^{\pi} = \frac{-2}{\pi k}\left[\cos(k\pi) - 1\right]$$

$$b_k = \begin{cases} 0 & \text{IF } k = 2n \text{, EVEN} \\ \dfrac{4}{\pi(2n+1)} & \text{IF } k = 2n+1 \text{, ODD} \end{cases}$$

So

$$\tilde{f}_{ODD,PER}(x) \sim \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin[(2n+1)x].$$

DIFFERENTIATION OF FOURIER SERIES

Suppose
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

We would like to have
$$f'(x) = \sum_{k=1}^{\infty} -k a_k \sin(kx) + k b_k \cos(kx)$$
wherever f is differentiable.

"DIFFERENTIATE TERM BY TERM"

TROUBLING EX

We just proved that for $0 < x < \pi$ we have
$$1 = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin[(2n+1)x]$$

BUT
$$0 = \frac{d}{dx}(1) \neq \sum_{n=0}^{\infty} \frac{4}{\pi} \cos[(2n+1)\cancel{x}]$$

DIVERGES for $0 < x < \pi$

So we need to impose some extra conditions on f to allow us to differentiate F.S. term by term.

IDEA To ensure FS of f' CONVS we need f' PW C^1.

THM ON DIFFERENTIATION OF F.S.

Suppose that $f:[-\pi,\pi]$ has a CTS, 2π-PERIODIC piecewise C^2 extension to $\mathbb{R}$.

If
$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

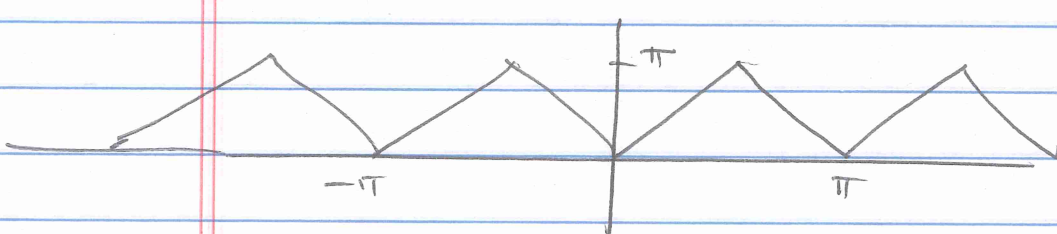
Then
$$f'(x) \sim \sum_{k=1}^{\infty} k b_k \cos kx - k a_k \sin kx$$

EXS

① Our previous example was not CTS so Thm does not apply.

② Let $f(x) = |x|$.

on $[-\pi,\pi]$:
$$f'(x) = \text{sign}(x)$$
$$= \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$



f has CTS 2π-periodic, PW C^2 ext^n to $\mathbb{R}$ ✓

UFS ✓
$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$$

AND SO
$$\text{sign}(x) \sim \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

which UFS ✓ is FS of sign(x).

CHANGE OF SCALE

LET $\quad f : [-L, L] \to \mathbb{R}.$

Define $\quad s : [-\pi, \pi] \to [-L, L]$ by

$$s(y) = \frac{L}{\pi} y$$

and set

$$F(y) = f(s(y)) = f\left(\frac{L}{\pi} y\right) = f(x) \qquad x = \frac{L}{\pi} y$$

So that $\quad F : [-\pi, \pi] \to \mathbb{R}.$

We have

$$F(y) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(ky) + b_k \sin(ky)$$

with

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos(ky)\, dy$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \sin(ky)\, dy.$$

$$y = \frac{\pi}{L} x$$

$\boxed{\text{So}}$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)$$

with

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \qquad dy = \frac{\pi}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) dx$$