LECTURE 7 — FOURIER SERIES

DEF Let $f, g : [-\pi, \pi] \to \mathbb{R}$ be CTS. (but not necessarily periodic)

Define the L^2 – INNER PRODUCT of f and g by

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

and the NORM of f by

$$\|f\| := \sqrt{\langle f, f \rangle}.$$

LEMMA The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ ~~satisfy~~ are PERIODIC on $[-\pi, \pi]$ and satisfy

$$\langle \cos kx, \cos lx \rangle = 0 = \langle \sin kx, \sin lx \rangle, \quad l \neq k, \; l, k \geq 0$$

$$\langle \cos kx, \sin lx \rangle \qquad \forall\, l, k \geq 0$$

$$\|1\| = \sqrt{2}, \quad \|\cos kx\| = 1 = \|\sin kx\| \quad \forall k > 0.$$

PF

(1) $$\langle \cos kx, \sin lx \rangle = \int_{-\pi}^{\pi} \underbrace{(\cos kx)(\sin lx)}_{\text{ODD}} dx = 0$$

$\quad$ EVEN $\times$ ODD

EVEN

2

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[(k+\ell)x] + \cos[(k-\ell)x]\, dx$$

$$= \begin{cases} \dfrac{1}{2\pi}\left[\dfrac{\sin(k+\ell)x}{k+\ell} + \dfrac{\sin(k-\ell)x}{k-\ell}\right]_{-\pi}^{\pi} = 0 & \text{IF } k \neq \ell \\[2ex] \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} 1\, dx = 1 & k = \ell. \end{cases}$$

SUPPOSE THAT

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

Then formally (without worrying whether series converge, etc):

$$\langle f(x), \cos(\ell x)\rangle = \frac{a_0}{2}\langle 1, \cos(\ell x)\rangle + \sum_{k=1}^{\infty} a_k \langle \cos(kx), \cos(\ell x)\rangle + b_k \langle \sin(kx), \cos(\ell x)\rangle$$

$$= \begin{cases} \frac{a_0}{2} \langle 1, 1 \rangle = a_0 & \text{IF } \ell = 0 \\ a_\ell & \text{IF } \ell \neq 0 \end{cases}$$

Similarly $\langle f(x), \sin(\ell x) \rangle = b_k$.

DEF The FOURIER SERIES of $f : [-\pi, \pi] \to \mathbb{R}$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

where

$$a_k = \langle f, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \qquad k = 0, 1, 2, \ldots$$

$$b_k = \langle f, \sin kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \qquad k = 1, 2, \ldots$$

NOTE

1. NO GAURANTEE That F.S. converges, or that if it does that ~~if~~ its value at x equals $f(x)$.

   HENCE NOTATION $\sim$ instead of $=$.

2. If The F.S. of f converges, it may NOT be CB or differentiable (term by term) even if f is.

EX S

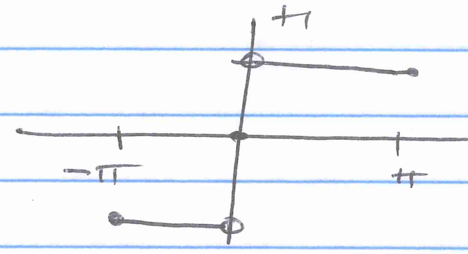
① [See Oliver P76] $f(x) = x.$

UG ✓

$$x \sim 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

$$= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx).$$

② $f(x) = \mathrm{sign}(x) = \begin{cases} 1 & \text{IF } x > 0 \\ 0 & \text{IF } x = 0 \\ -1 & \text{IF } x < 0 \end{cases}$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{sign}(x)\, dx = 0 \quad \text{by Symmetry.}$$

$$\cancel{b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{sign}(x) \sin(kx)}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{sign}(x) \cos(kx)\, dx = 0$$

ODD x EVEN = ODD

BUT

$$b_k = \frac{1}{\pi} \int_{-\pi}^{0} -\sin kx\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin kx\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin(kx)\, dx = \frac{2}{\pi} \left[\frac{-\cos(kx)}{k} \right]_{0}^{\pi} \qquad k = 1, 2, \ldots$$

$$b_k = \frac{2}{\pi}\left[\frac{1}{k} - \frac{\cos k\pi}{k}\right]$$

$$= \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{4}{\pi k} & \text{if } k \text{ odd} \quad (k = 2n+1) \end{cases}$$

So

$$\operatorname{sign}(x) \sim \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)} \sin[(2n+1)x]$$

<u>DEF</u> The n-th PARTIAL SUM of the FS of f is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=0}^{n} [a_k \cos kx + b_k \sin kx]$$

We say that the FS of f <u>CONVERGES at x</u> if

$$\lim_{n \to \infty} S_n(x) = \hat{f}(x) \quad \exists.$$

WE WOULD LIKE $f(x) = \hat{f}(x) \quad \forall x \in [-\pi, \pi]$

But there is no gaurantee this holds.

PERIODIC EXTENSIONS

$\boxed{\text{FOURIER SERIES OF } f}$

We start with $f : [-\pi, \pi] \longrightarrow \mathbb{R}$.

We obtain

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

NOTICE

$S_n : \mathbb{R} \longrightarrow \mathbb{R}$ is periodic:

$$S_n(x+2\pi) = S_n(x).$$

So if $\tilde{f}(x) = \lim_{n \to \infty} S_n(x)$ $\exists$ we must have

$$\tilde{f}(x+2\pi) = \tilde{f}(x), \qquad \tilde{f} : \mathbb{R} \to \mathbb{R}$$

$\boxed{\text{PERIODIC EXTENSION OF } f}$

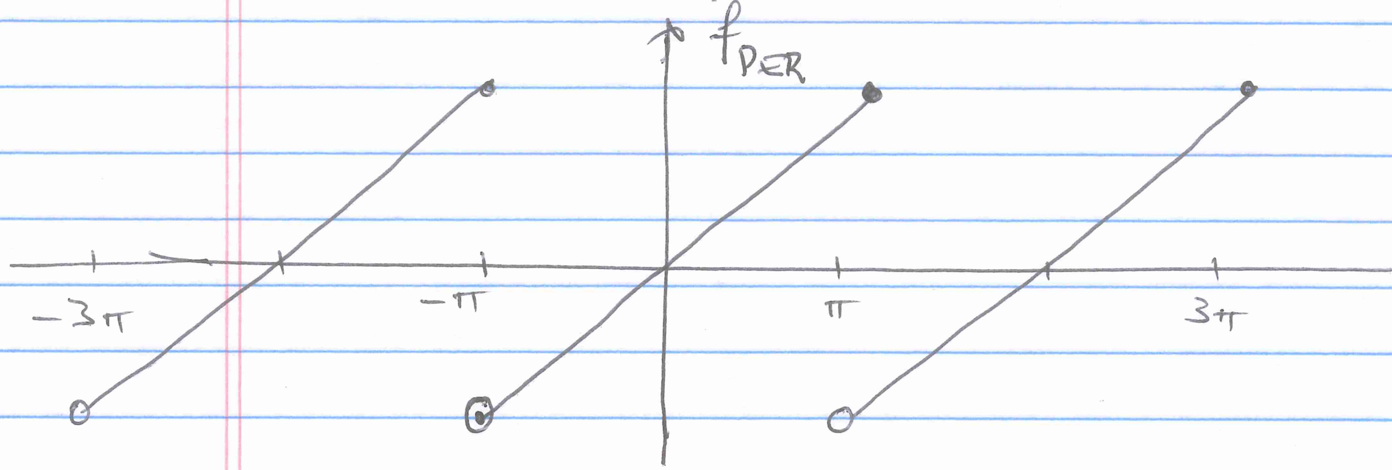
Given $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ we can define

$\tilde{f}_{PER} : \mathbb{R} \to \mathbb{R}$ periodic of period 2π so that

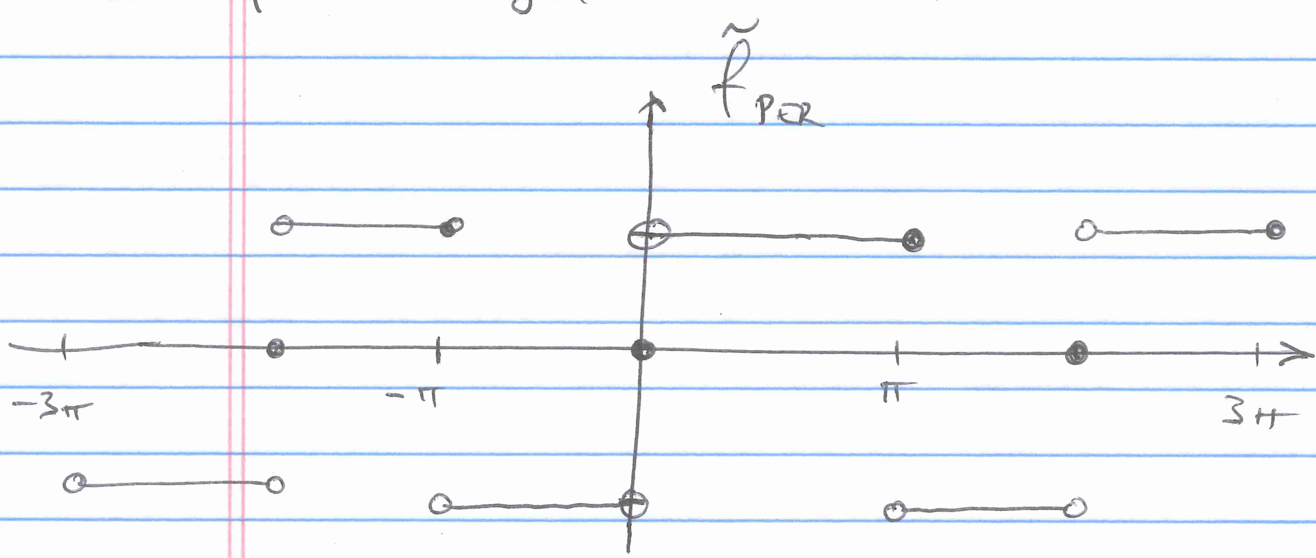
$$\tilde{f}_{PER}(x) = f(x) \qquad \forall x \in [-\pi, \pi].$$

MAYBE $\tilde{f} = \tilde{f}_{PER}$ for nice enough f?

<u>EX</u> ① $f(x) = x$ on $[-\pi, \pi]$ has



② $f(x) = \text{sign}(x)$ on $[-\pi, \pi]$ has



DEF $f : [a,b] \to \mathbb{R}$ is PIECEWISE CONTINUOUS (PWCB) if

(a) f is defined and continuous on $[a,b]$ except possibly at a finite # of points

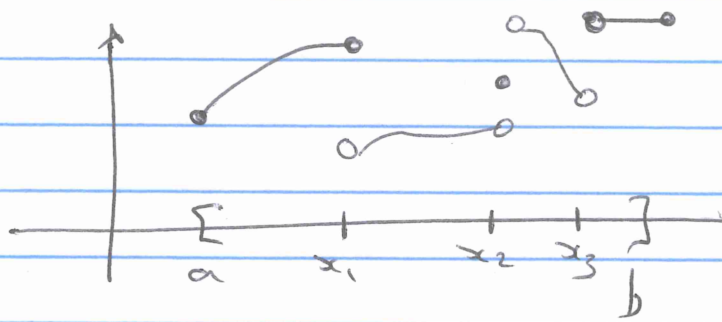
$$a \le x_1 < x_2 < \dots < x_n \le b,$$

(b) At each discontinuity x_k we require

$$\lim_{x \to x_k^-} f(x) =: f(x_k^-) \ \exists$$

$$\lim_{x \to x_k^+} f(x) =: f(x_k^+) \ \exists.$$

EX



DEF $f : [a,b] \to \mathbb{R}$ is PIECEWISE C^1 if

(a) f is defined, CTS, C^1 (continuously differentiable) on $[a,b]$ except possibly at finite # pts

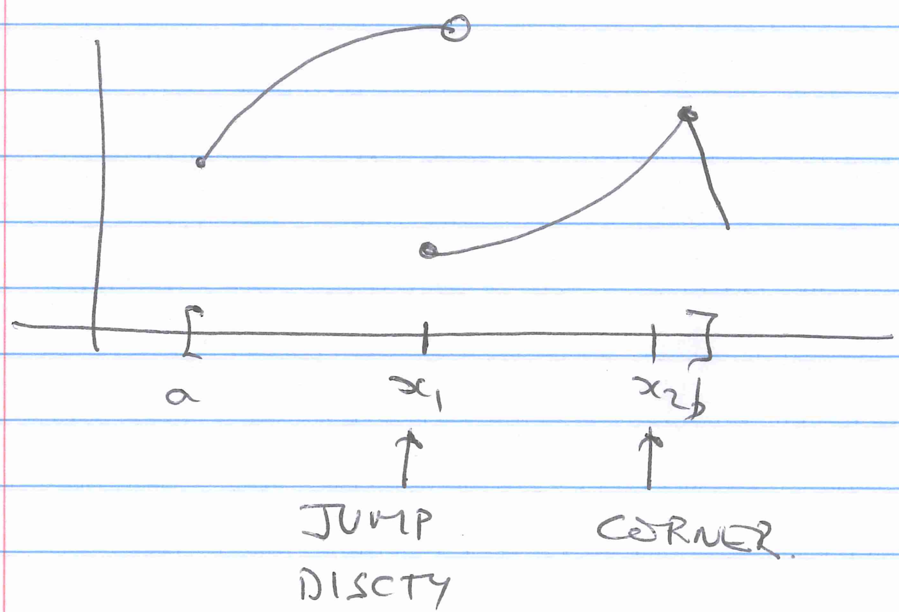
$$a \le x_1 < x_2 < \dots < x_n \le b$$

(b) At each x_k we require

$$\lim_{x \to x_k^\pm} f(x) =: f(x_k^\pm) \ \exists$$

$$\lim_{x \to x_k^\pm} f'(x) =: f'(x_k^\pm) \ \exists.$$

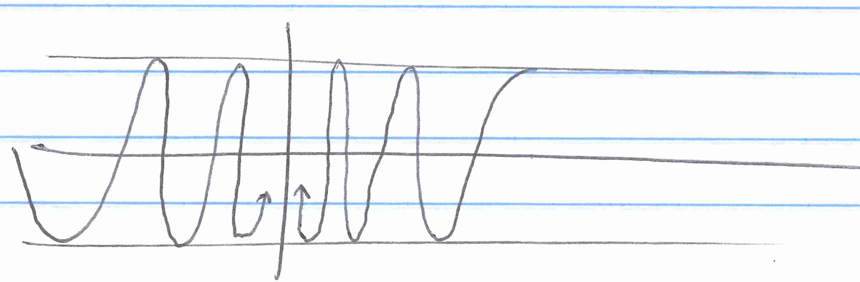
<u>EX</u>



<u>NOTE</u>

(1) $f(x) = \sin\left(\frac{1}{x}\right)$ on $[-1, 1]$ is <u>NOT</u> PW CTS

as $\lim_{x \to 0^\pm} \sin\left(\frac{1}{x}\right)$ DNE



2 $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{IF } x \neq 0 \\ 0 & \text{IF } x = 0 \end{cases}$

is CTS on $[-1, 1]$ but NOT PW C^1

as

$$f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x} \cos x, \qquad x \neq 0$$

has $\lim_{x \to 0^\pm} f'(x)$ DNE.