LECTURE 6 INTRODUCTION TO FOURIER SERIES

FOURIER, $\simeq$ 1800, while studying heat flow suggested that

"Every" function can be expressed as an infinite series of sin and cos functions.
— "Fourier Series".

APPLICATION include
- Music
- Signal and Image Processing
- Optics, Electronics
- Communications Technology

LED to development of
- ANALYSIS (MATH 3310, 4301, 4302, ...)
- LINEAR ALGEBRA

MOTIVATION FROM ODES AND LINEAR ALGEBRA

(A) SCALAR ODE : $u = u(t)$ $\quad u : \mathbb{R} \to \mathbb{R}$.

$$\begin{cases} \dfrac{du}{dt} = \lambda u \\ u(0) = u_0 \end{cases} \qquad (1)$$

Has solution $u(t) = u_0 e^{\lambda t}$

B LINEAR SYSTEM OF ODES

$\vec{u} = \vec{u}(t)$, $\quad \vec{u} : \mathbb{R} \to \mathbb{R}^n$

$$\begin{cases} \dfrac{d\vec{u}}{dt} = A\vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

1ST ORDER
A is n×n MATRIX (3)

$\boxed{\text{EX}}$ Given second order ODE for $v = v(t)$

$$\begin{cases} v'' + av' + bv = 0 \\ v(0) = v_0 \\ v'(0) = w_0 \end{cases}$$

Damped Harmonic Oscillator

Let $w = v'$

So

$$w' + aw + bv = 0$$
$$v' = w$$

Gives $\begin{bmatrix} v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$

Let $\vec{u}(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ $\quad \vec{u}_0 = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$

Then we get

$$\begin{cases} \dfrac{d\vec{u}}{dt} = A\vec{u} \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

NOW SOLVE LINEAR SYSTEMS OF ODEs USING LINEAR ALGEBRA
LATER SOLVE LINEAR PDEs USING FOURIER THEORY

MOTIVATED BY (A), to solve (2) ~~gue~~ look for solutions of form

$$\vec{u}(t) = e^{\lambda t}\vec{v} \qquad \vec{v} = \begin{bmatrix} v_1 \\ | \\ v_n \end{bmatrix} \text{ is time indept}$$

Then

$$\frac{d\vec{u}}{dt} = A\vec{u} \qquad \text{gives}$$

$$\lambda e^{\lambda t}\vec{v} = A e^{\lambda t}\vec{v}$$

OR

$$\boxed{A\vec{v} = \lambda\vec{v}} \qquad \text{LINEAR ALG PROBLEM!!}$$

So $(\lambda, \vec{v})$ is an eigenpair for matrix A

THM [See Appendix]

Let A be a real $n \times n$ matrix that is symmetric, $A^T = \overline{A}$. Then

1. All evalues of A are real
2. Evectors corresponding to distinct evalues are orthogonal
3. $\exists$ ONB of $\mathbb{R}^n$ consisting of evectors of A.

GIVEN THM

Let $\{\vec{v}_1, \dots \vec{v}_n\}$ be an ONB of evectors of A, so That

$$A\vec{v}_j = \lambda_j \vec{v}_j \qquad j = 1 - n.$$

So for any $\vec{w} \in \mathbb{R}^n$, $\exists\ c_1, \dots c_n \in \mathbb{R}$:

$$\boxed{\vec{w} = \sum_{j=1}^{n} c_j \vec{v}_j} \quad (3)$$

REWRITE (3) : IF $V = [\vec{v}_1, \dots \vec{v}_n]$, $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Then (3) is

$$\boxed{\vec{w} = V\vec{c}} \quad (3')$$

In general to find coeffs c_j, we need to solve linear system (3'), say using Gaussian Elimination. (YUK!)

BUT since $\{\vec{v}_1, \dots \vec{v}_n\}$ are ONB we have

CLAIM

$$\boxed{c_j = \vec{w} \cdot v_j} \quad (4) \qquad \text{SUPER EASY!}$$

PF

$$\vec{w} = \sum_{k=1}^{n} c_k \vec{v}_k$$

$$\vec{w} \cdot \vec{v}_j = \left(\sum_k c_k \vec{v}_k\right) \cdot \vec{v}_j$$

$$= \sum_k c_k (\vec{v}_k \cdot \vec{v}_j)$$

$$= \sum_k c_k \delta_{kj} = c_j$$

D.

BACK TO

$$\begin{cases} \dfrac{d\vec{u}}{dt} = A\vec{u} \\ \vec{u}(0) = \vec{u}_0. \end{cases}$$

SUPPOSE $A^T = A$.

Let $A\vec{v}_j = \lambda_j \vec{v}_j$ as in Thm

Then we know

$$\vec{u}_j(t) = e^{\lambda_j t} \vec{v}_j$$ solves $\frac{d\vec{u}}{dt} = A\vec{u}$.

Since ODE is linear so does

$$\boxed{\vec{u}(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \vec{v}_j}$$ for $c_j \in \mathbb{R}$

By IC:

$$\vec{u}_0 = \vec{u}(0) = \sum_{j=1}^{n} c_j \vec{v}_j$$

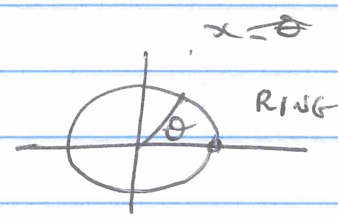
and by ~~CLAIM~~ THM since $\{\vec{v}_1 - \vec{v}_n\}$ is ONB for $\mathbb{R}^n$ we can solve for ANY $\vec{u}_0$, with

$$\boxed{c_j = \vec{u}_0 \cdot \vec{v}_j}$$ by Claim

SOLN

$$\boxed{\vec{u}(t) = \sum_{j=1}^{n} [\vec{u}_0 \cdot \vec{v}_j] e^{\lambda_j t} \vec{v}_j}$$

PDE CASE: EX of HEATED RING



SIMPLEST EX $\quad u = u(t, x)$

$$\begin{cases} u_t = u_{xx} & x \in [-\pi, \pi],\ t \geq 0 \\ u(t, -\pi) = u(t, \pi) & \text{PERIODIC} \\ u_x(t, -\pi) = u_x(t, \pi) & \text{BCs} \\ u(0, x) = f(x) & \text{INITIAL HEAT DISTRIBUTION} \end{cases}$$

DISCRETIZE $[-\pi, \pi]$ using n points $x_1, \ldots, x_n$:

$$x_k = -\pi + (k-1)\Delta x \qquad \Delta x = \frac{2\pi}{n}.$$

and discretize a function $f : [-\pi, \pi] \to \mathbb{R}$ by a vector

$$\vec{v} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} \in \mathbb{R}^n.$$

THE GRAND ANALOGY

ODEs + LINEAR ALGEBRA		PDEs + FOURIER THEORY
VECTOR SPACE, $\mathbb{R}^n$		VECTOR SPACE OF FUNCTIONS FROM $[-\pi, \pi]$ to $\mathbb{R}$
VECTOR $\vec{v} \in \mathbb{R}^n$		FUNCTION $f : [-\pi, \pi] \to \mathbb{R}$
INNER PRODUCT	$\langle \vec{v}, \vec{w} \rangle = \sum_{k=1}^{n} v_k w_k$	L^2-INNER PRODUCT $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$
ODE	$\frac{d\vec{u}}{dt} = A\vec{u} = L[\vec{u}]$	PDE-BVP $\frac{\partial u}{\partial t} = \mathcal{L}[u] = u_{xx}$ $u(t, -\pi) = u(t, \pi)$ $u_x(t, -\pi) = u_x(t, \pi)$

ODEs + LINEAR ALGEBRA	PDEs + FOURIER THEORY
IC $\vec{u}(0) = \vec{u}_0$	$u(0,x) = f(x)$
E-VALUES + E-VECTORS	E-VALUES + E-FUNCTIONS
$A\vec{v}_k = \lambda_k \vec{v}_k$	$\mathcal{L}[\cos kx] = -k^2 \cos kx$ $\mathcal{L}[\sin kx] = -k^2 \sin kx$ IF $k = 0, 1, 2..$ then $\cos kx$, $\sin kx$ satisfy the periodic BCs.
ONB of E-VECTORS OF A $\{\vec{v}_1, \ldots \vec{v}_n\}$	FOURIER SERIES BASIS (ON!!) $\{1, \cos x, \cos(2x), \cos(3x) \ldots$ $\sin x, \sin(2x), \sin(3x) \ldots\}$
REPRESENTATION OF $\vec{w}$ IN ONB $\forall \vec{w} \in \mathbb{R}^n \; \exists c_k :$ $\vec{w} = \sum_{k=1}^{n} c_k \vec{v}_k$	FOURIER SERIES OF f "$\forall$" $f : [-\pi, \pi] \to \mathbb{R}$, periodic $\exists a_k, b_k :$ $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$
$c_k = \vec{w} \cdot \vec{v}_k$	$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$ $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$
GS to ODE $\vec{u}(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t} \vec{v}_k$	GS TO PDE $u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-k^2 t}[a_k \cos kx + b_k \sin kx]$

ISSUE INFINITE SERIES may not converge.

KEY ANALYTICAL QUESTIONS

1. For which pairs of sequences $\{a_k\}$, $\{b_k\}$ does the corresponding Fourier Series converge

2. What sorts of functions can be represented by a convergent Fourier Series

3. Can we differentiate Fourier series term by term?

4. Does the Fourier Series solution actually solve the Heat eqn BVP/IVP?