EXTERNAL FORCING

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(t,x) \\ \\ u(0,x) = 0 \\ u_t(0,x) = 0 \end{cases}$$

External force Applied at time t + position x

FOR SIMPLICITY

Wave is set in motion / forced by F.
So no need for non-zero ICs.

As in pf of Thm I set

$$\xi = x - ct \qquad \eta = x + ct$$

So

$$x = \frac{\eta + \xi}{2} \qquad t = \frac{\eta - \xi}{2c}$$

Set

$$v(\xi,\eta) = u(t,x) = u\left(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2}\right)$$

By Chain Rule as in Thm I

$$v_{\xi\eta} = -\frac{1}{4c^2}\left(u_{tt} - c^2 u_{xx}\right) = -\frac{1}{4c^2} F\left(\frac{\eta-\xi}{2c}, \frac{\eta+\xi}{2}\right)$$

By FTC, integration wrt η gives

$$\frac{\partial v}{\partial \xi}(\xi,\eta) - \frac{\partial v}{\partial \xi}(\xi,\xi) = -\frac{1}{4c^2}\int_{Z=\xi}^{Z=\eta} F\left(\frac{Z-\xi}{2c}, \frac{Z+\xi}{2}\right) dZ \tag{+}$$

Also by CR $V\boxed{I}$ ✓

$$\frac{\partial v}{\partial \xi}(\xi,\xi) = \frac{1}{2c}\frac{\partial u}{\partial t}(0,\xi) + \frac{1}{2}\frac{\partial u}{\partial x}(0,\xi)$$

$$= 0 \quad \text{by ICs.}$$

So (†) is of form

$$\frac{\partial v}{\partial \xi}(\xi,\eta) = G(\xi,\eta) \qquad (\ddagger)$$

where

$$G(\xi,\eta) = -\frac{1}{4c^2}\int_{z=\xi}^{z=\eta} F\left(\frac{z-\xi}{2c}, \frac{z+\xi}{2}\right) dz.$$

Fix η

Integrating $(\ddagger)$ wrt ξ over $\xi \le X \le \eta$,

$$v(\eta,\eta) - v(\xi,\eta) = \int_{X=\xi}^{X=\eta} \frac{\partial v}{\partial \xi}(X,\eta)\, dX$$

$$= \int_{X=\xi}^{X=\eta} G(X,\eta)\, dX$$

Since $v(\eta,\eta) = u(0,\eta) = 0$ by IC we

get

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GIVEN FIXED $(\xi,\eta) = (x-ct, x+ct)$, ie fixed (t,x) we have

$$v(\xi,\eta) = \frac{1}{4c^2}\int_{X=\xi}^{X=\eta}\int_{Z=X}^{Z=\eta} F\left(\frac{Z-X}{2c}, \frac{Z+X}{2}\right) dZ\, dX$$

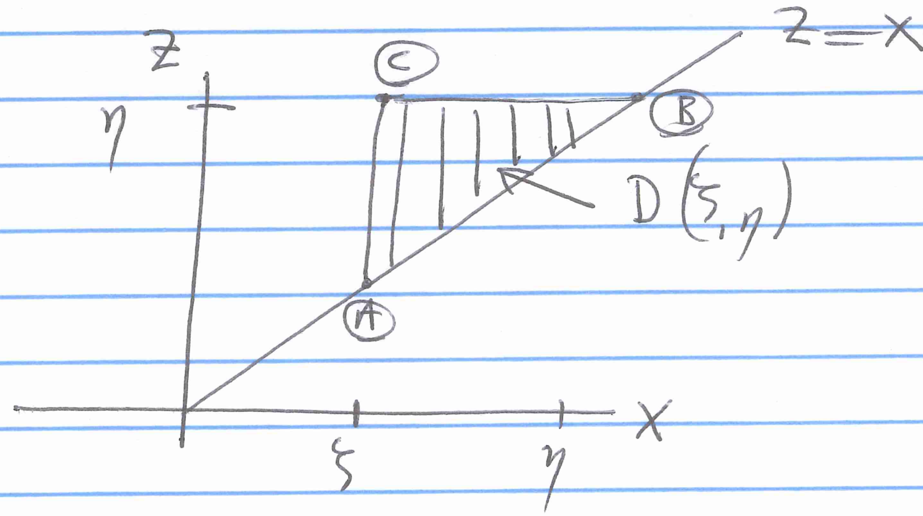
OR

$$u(t,x) = v(x-ct, x+ct)$$

$$= \frac{1}{4c^2}\int_{X=x-ct}^{X=x+ct}\int_{Z=X}^{Z=x+ct} F\left(\frac{Z-X}{2c}, \frac{Z+X}{2}\right) dZ\, dX$$

$\xi \le X \le \eta$

$X \le Z \le \eta$



Change of Variables in Integral:

$$s = \frac{Z-X}{2c} \qquad y = \frac{Z+X}{2}$$

ie $\quad X = y - cs \qquad Z = y + cs$

(A) $(X, Z) = (\xi, \xi) = (x - ct,\ x - ct)$

Gives $\quad X = \xi = Z \implies s = 0$

And $\quad y = X = x - ct$

So (A') : $\quad (s, y) = (0,\ x - ct)$

(B) $(X, Z) = (\eta, \eta) = (x + ct,\ x + ct)$

Gives $\quad X = \eta = Z \implies s = 0$

And $\quad y = X = x + ct$

So (B') : $\quad (s, y) = (0,\ x + ct)$

$\cancel{0 \le s \le t}$

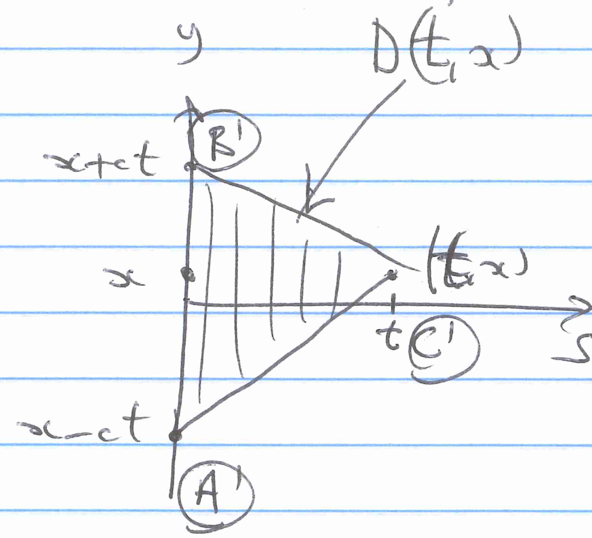
$\cancel{x - c(t-s) \le y \le x + c(t-s)}$

(C) $(X, Z) = (\xi, \eta) = (x - ct,\ x + ct)$

Gives $\quad y = \dfrac{Z + X}{2} = x$

$\qquad s = \dfrac{Z - X}{2c} = t$

So (C') $\quad (s, y) = (t, x)$



$D(t,x)$ is

$$0 \le s \le t$$

$$x - c(t-s) \le y \le x + c(t-s)$$

So

$$\boxed{u(t,x) = \frac{1}{2c} \int_{s=0}^{s=t} \int_{y=x-c(t-s)}^{y=x+c(t-s)} f(s,y)\, dy\, ds}$$

$\boxed{\text{SUMMARY}}$

The solution of IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(t,x) & t>0,\ x\in\mathbb{R} \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

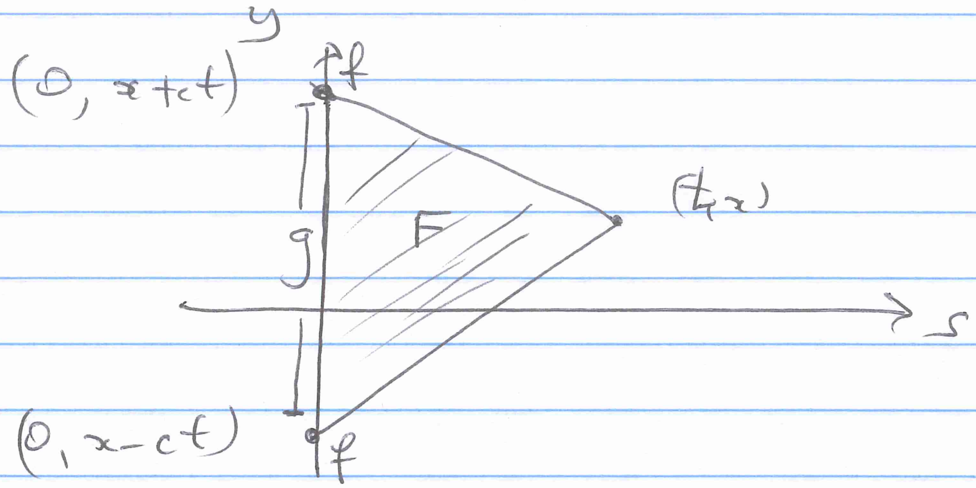
is

$$u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c} \int_{y=x-ct}^{y=x+ct} g(y)\, dy$$

$$+ \frac{1}{2c} \int_{s=0}^{s=t} \int_{y=x-c(t-s)}^{y=x+c(t-s)} F(s,y)\, dy\, ds.$$

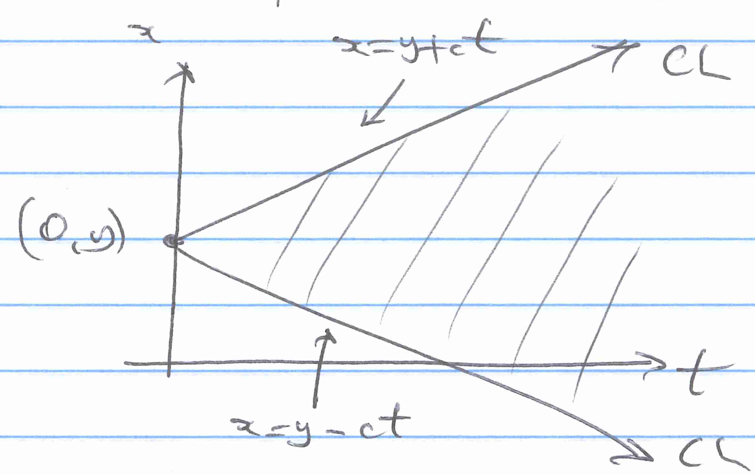
DOMAIN OF DEPENDENCE OF POINT (t,x)

- This is set of all (s,y) so that solution at (t,x) depends on values of f, g, F at (s,y)



DOMAIN OF INFLUENCE OF POINT $(0,y)$ [CASE $F \equiv 0$]

- This is set of all (t,x) so that values of f, g at $(0,y)$ influence value of u at (t,x).



ⓐ Value of f at $(0,y)$ affects value of u on 2 CL's

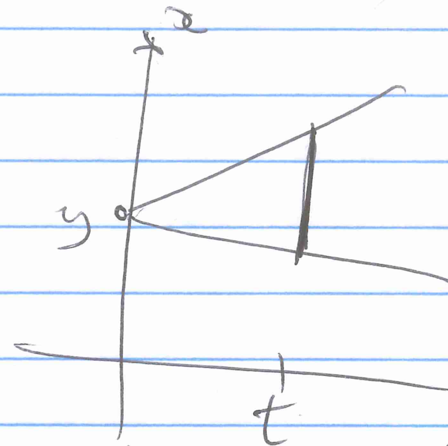
ⓑ Value of g at $(0,y)$ affects value of u in Shaded Region

Reason for (1)

$$u(t,x) = \int_{y=x-ct}^{y=x+ct} g(y)\,dy$$

$$x - ct < y < x + ct$$

$$\Longleftrightarrow \quad y - ct < x < y + ct$$



So

Given $t > 0$

Value of g at y contributes to $u(t,x)$ integral when $x \in (y-ct, y+ct]$

EX

$$\begin{cases} u_{tt} - u_{xx} = \sin(\omega t)\sin x = F(t,x) \\ u(0,x) = 0 \\ u_t(0,x) = 0 \end{cases}$$

SOLN

$$u(t,x) = \frac{1}{2}\int_{s=0}^{s=t}\int_{y=x-t+s}^{y=x+t-s} \sin(\omega s)\sin y \,dy\,ds$$

$$= \frac{1}{2}\int_{s=0}^{s=t} \sin(\omega s)\left[\cos(x-t+s) - \cos(x+t-s)\right] ds$$

~~VERIF~~ $\cancel{\left\{ \sin(\omega t) - \omega s \right.}$

U II ✓

$$u(t,x) = \begin{cases} \dfrac{\sin(\omega t) - \omega \sin t}{1-\omega^2} \sin x & \omega \neq 1 \\[2ex] \dfrac{\sin t - t\cos t}{2} \sin x & \omega = 1. \end{cases}$$

When $\omega \neq 1$, $|u(t,x)| \leq \dfrac{1}{1-\omega}$ $\forall (t,x)$

BOUNDED

But when $\omega = 1$ $|u(t,x)| \to \infty$ as $t \to \infty$.
(provided $\sin x \neq 0$)

– RESONANCE.