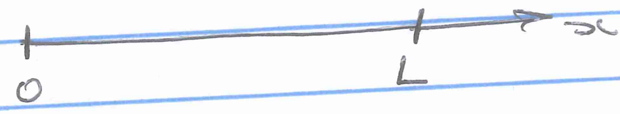
LECTURE 5 THE WAVE EQUATION ON $\mathbb{R}$ AND d'ALEMBERT'S FORMULA

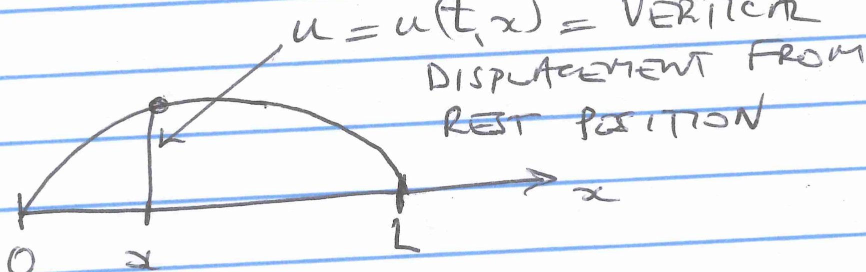
DERIVATION of WAVE EQN for Transverse Vibrations of a violin string.

① STRING AT REST



Let $\rho_0(x)$ = LINEAR DENSITY
= Mass per unit length along string

② STRING IN MOTION AT FIXED t:

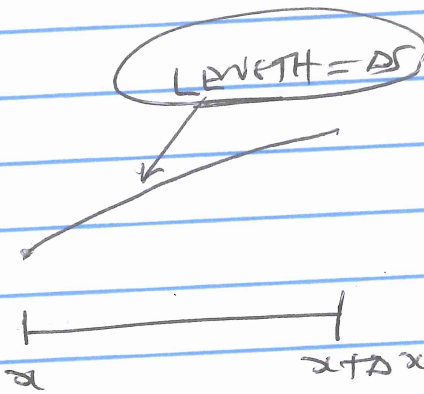


ASSUME PARTICLE ON STRING MOVES TRANSVERSE TO x-AXIS.

Let $\rho = \rho(t,x)$ = LINEAR DENSITY of String in Motion

SINCE LENGTH ② > LENGTH ①

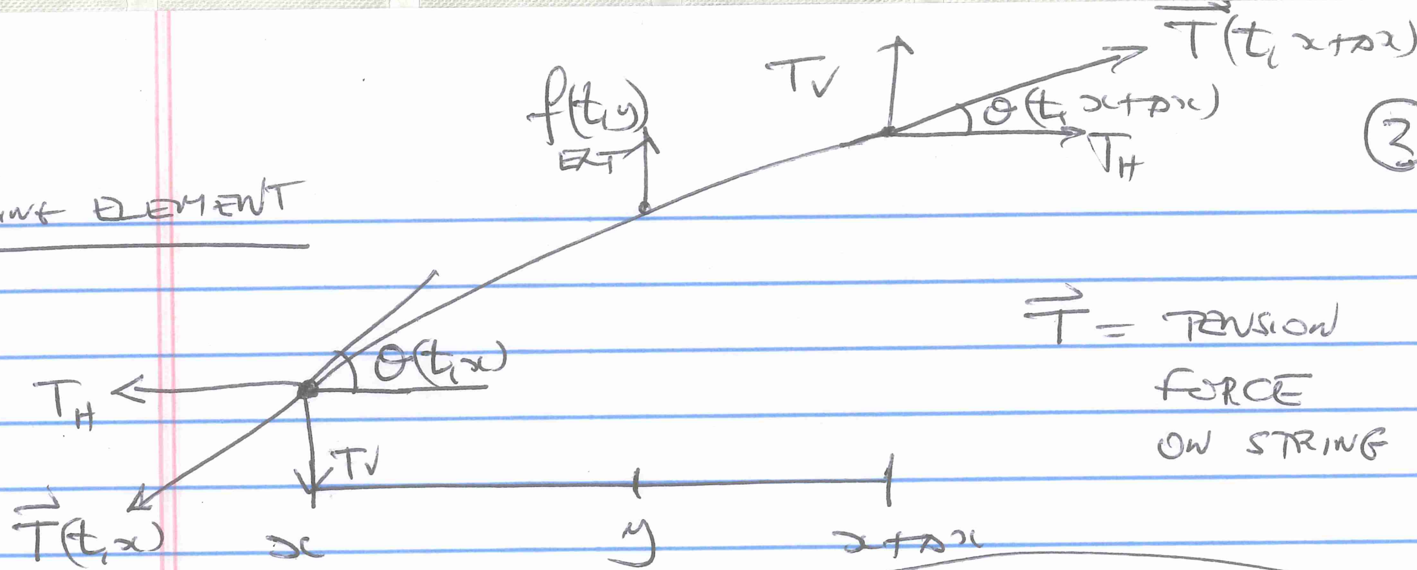
EXPECT $\rho(t,x) < \rho_0(x)$



CONSERVATION OF MASS:

MASS OVER $[x, x+\Delta x] = \rho_0(x)\,\Delta x = \rho(t,x)\,\Delta s$ ①

STRING ELEMENT



$\vec{T}$ = TENSION FORCE ON STRING

$f_{EXT}(t, y)$ = Vert. Force per unit length due to external loads eg PLUCKING

HORIZONTAL FORCE BALANCE

Since motion is entirely vertical

$$|T_H(t, x)| = |T_H(t, x+\Delta x)|$$

ie

$$|\vec{T}|(t, x) \cos\theta(t, x) = |\vec{T}|(t, x+\Delta x) \cos\theta(t, x+\Delta x)$$

OR

$$\frac{\partial}{\partial x}\left[|\vec{T}| \cos\theta \right] = 0$$

So

$$|\vec{T}| \cos\theta = K(t) = \text{CONST IN } x \quad (2)$$

$$= \text{HORIZONTAL TENSION}$$

VERTICAL FORCE BALANCE

Total Vertical Force is

$$F_{VERT} = T_V(t, x+\Delta x) - T_V(t, x) + F_{EXT} \quad (3)$$

Here

$$T_V(t,x) = |\vec{T}| \sin\theta$$

$$= K(t) \tan\theta \quad \text{by } 2$$

$$= K(t) \frac{\partial u}{\partial x}$$

SLOPE

So

$$T_V(t, x+\Delta x) - T_V(t,x) \overset{FTC}{=} \int_x^{x+\Delta x} K(t) \frac{\partial^2 u}{\partial x^2}(t,y)\, dy$$

NEXT

$$F_{EXT} = \int_x^{x+\Delta x} \rho(t,y) f_{EXT}(t,y)\, ds$$

$$\overset{1}{=} \int_x^{x+\Delta x} \rho_0(y) f_{EXT}(t,y)\, dy$$

So by 3 and "$F_{VERT} = m\, a_{VERT}$":

$$\int_x^{x+\Delta x} \left[K(t) \frac{\partial^2 u}{\partial x^2}(t,y) + \rho_0(y) f_{EXT}(t,y)\right] dy$$

$$= \int_x^{x+\Delta x} \rho(t,y) \frac{\partial^2 u}{\partial t^2}(t,y)\, ds$$

$$\overset{1}{=} \int_x^{x+\Delta x} \rho_0(y) \frac{\partial^2 u}{\partial t^2}(t,y)\, dy$$

(4)

So

$$\boxed{u_{tt} - \frac{K(t)}{\rho_0(y)}\, u_{xx} = f_{EXT}}$$

LET $c = c(t,y) = \sqrt{\dfrac{K(t)}{\rho_0(y)}}$ = WAVE SPEED

Then

$$\boxed{u_{tt} - c\, u_{xx} = f_{EXT}} \qquad (5)$$

WAVE EQN

NOTE

(a) STRING HOMOGENEOUS means $\rho_0(y) =$ CONST

(b) STRING ELASTIC means $K(t) =$ CONST

– HORIZONTAL TENSION same as at rest.

If (a), (b) hold Then $c =$ CONST ~~too~~.

— o —

~~PARTIAL~~

For Now solve IVP on $\mathbb{R}$:

$$\begin{cases} u_{tt} - c\, u_{xx} = h(t,x) & \text{ON } t \geq 0,\ x \in \mathbb{R} \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases} \qquad (6)$$

DERIVATION OF d'ALEMBERT'S SOLUTION [$c = $ CONST]

Wave Operator

$$\Box = \partial_t^2 - c^2 \partial_x^2$$

$$= (\partial_t - c\partial_x)(\partial_t + c\partial_x) \qquad (7)$$

$$= (\partial_t + c\partial_x)(\partial_t - c\partial_x) \qquad (8)$$

SUPPOSE $u = u(t,x)$ SOLVES $\begin{cases} \text{TRANSPORT EQN} \\ \text{1-WAY WAVE EQN} \end{cases}$

$$u_t + c u_x = 0. \qquad (9)$$

Then by (7)

$$\Box u = (\partial_t - c\partial_x)(\partial_t + c\partial_x)(u)$$

$$= (\partial_t - c\partial_x)\, 0 = 0.$$

So u solves wave eqn.

We know sol^n of (9) is of form

$$u(t,x) = p(x - ct)$$

for some f^n $p = p(\xi)$.

THM I

Every sol^n of wave eqn $u_{tt} - c^2 u_{xx} = 0$ is of form

$$\boxed{u(t,x) = p(x-ct) + q(x+ct)} \quad (10)$$

for some functions $p = p(\xi)$, $q = q(\eta)$

i.e. u = SUPERPOSITION of RIGHT and LEFT travelling waves.

PF

By arguement above, (10) solves wave eqn.

But we still have to show EVERY sol^n has this form

Let $\xi = x - ct$, $\eta = x + ct$ CHARACTERISTIC VARIABLES

So

$$x = \frac{\eta + \xi}{2} \qquad t = \frac{\eta - \xi}{2c}$$

Define

$v = v(\xi, \eta)$ by $v(\xi, \eta) = u(t, x)$

So

$$\boxed{v(\xi, \eta) := u\left(\frac{\eta - \xi}{2c}, \frac{\eta + \xi}{2}\right)}$$

Then

$$u(t,x) = v(x - ct, x + ct) \quad (11)$$

CLAIM

$$\Box u = 0 \iff v_{\xi\eta} = 0.$$

PF By (11)

$$u_t = -c\, v_\xi + c\, v_\eta$$

$$u_{tt} = \left[(-c)^2 v_{\xi\xi} + -c^2 v_{\xi\eta}\right] + \left[c^2 v_{\eta\eta} \mp c^2 v_{\eta\xi}\right]$$

$$u_{tt} = c^2\left[v_{\xi\xi} + v_{\eta\eta} - 2v_{\xi\eta}\right]$$

AND $u_{xx} = v_{\xi\xi} + v_{\eta\eta} + 2v_{\xi\eta}$

So

$$\Box u = u_{tt} - c^2 u_{xx} = -4c^2 v_{\xi\eta} \qquad \blacksquare$$

FINALLY Let $w = \dfrac{\partial v}{\partial \eta}$

Then $0 = \dfrac{\partial^2 v}{\partial \xi \partial \eta} = \dfrac{\partial w}{\partial \xi}$

So $w = w(\eta)$ holds

$$\frac{\partial v}{\partial \eta} = w(\eta)$$

So $v(\xi,\eta) = \int w(\eta)\, d\eta + p(\xi) =: q(\eta) + p(\xi)$ $\blacksquare$

NEXT STEP

SOLVE IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases} \qquad (12)$$

f, g GIVEN.

By Thm I we know sol^n is of form

$$u(t,x) = p(x-ct) + q(x+ct)$$

GOAL Use IC to find p, q in terms of f, g.

Well

$$f(x) \overset{*}{=} p(x) + q(x) \Rightarrow f' = p' + q'$$

$$g(x) = -c\,p'(x) + c\,q'(x) \Rightarrow g = -cp' + cq'$$

So

$$cf' + g = cp' + cq' - cp' + cq' = 2cq'$$

OR

$$q' = \frac{1}{2}\left(f' + \frac{1}{c} g\right)$$

So

$$q(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(z)\,dz + \underset{\text{CONST}}{a}$$

By $*$

$$p(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(z)\,dz - a$$

So

THM II

Soln of IVP (12) is

$$\boxed{u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz} \quad (13)$$

EXS

(1) $g(x) = 0$. $f(x) = \dfrac{1}{1+x^2}$ LOCALIZED PULSE

ie

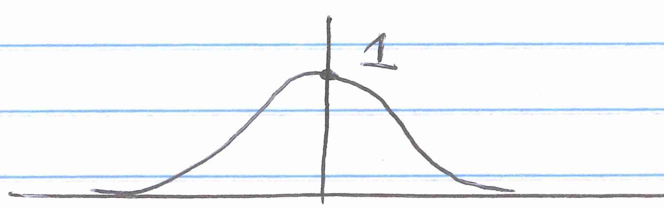
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = \dfrac{1}{1+x^2} \\ u_t(0,x) = 0 \end{cases}$$

"Since $g=0$, Wave doesn't know which way to go. So goes both way"

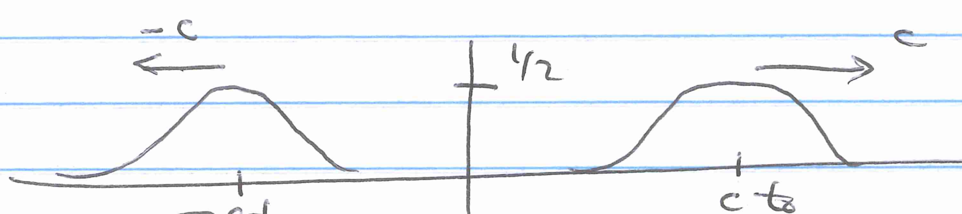
Soln

$$u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x-ct)^2} + \frac{1}{1+(x+ct)^2}\right]$$

$t=0$



$t=t_0$



Initial displacement splits into 2 ways.
- One moves left @ speed c, one right @ speed c
- Amplitude of each wave is $\frac{1}{2}$ that of initial condition.

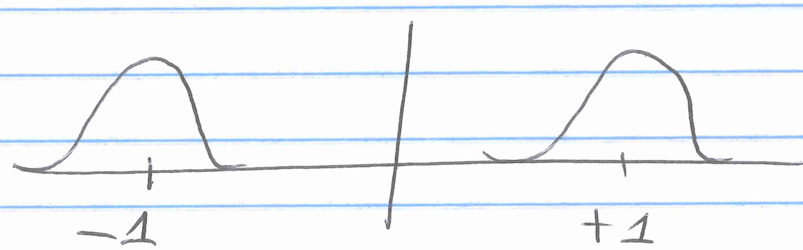
② $g(x) = 0$, $f(x) = \frac{1}{1+(x-1)^2} + \frac{1}{1+(x+1)^2}$

$$u(t,x) = \frac{1}{2}\left[\frac{1}{1+(x-ct-1)^2} + \frac{1}{1+(x+ct-1)^2} + \frac{1}{1+(x-ct+1)^2} + \frac{1}{(1+x+ct)^2}\right]$$

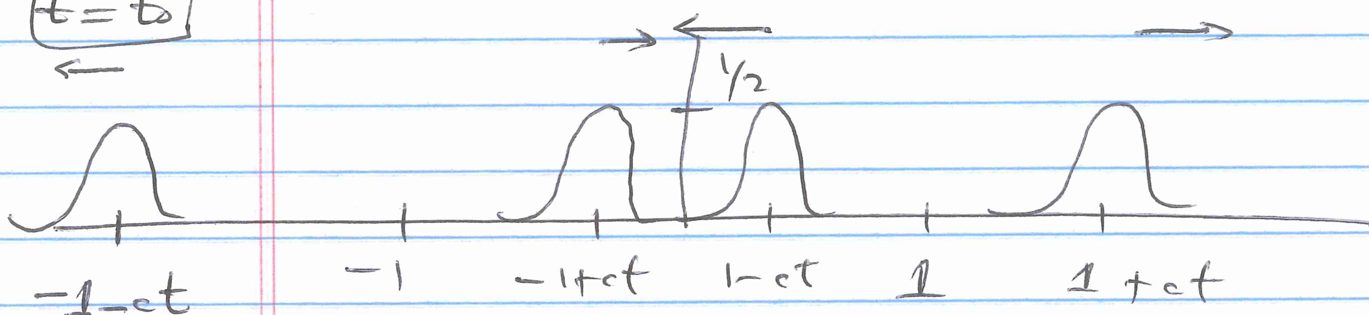
Two pulses split into 4 pulses.

$\boxed{t=0}$

PAIR OF LOCALIZED PULSES



$\boxed{t=t_0}$



These two will pass through each other.

(3) $g(x)=0, \quad f(x)=\cos x$

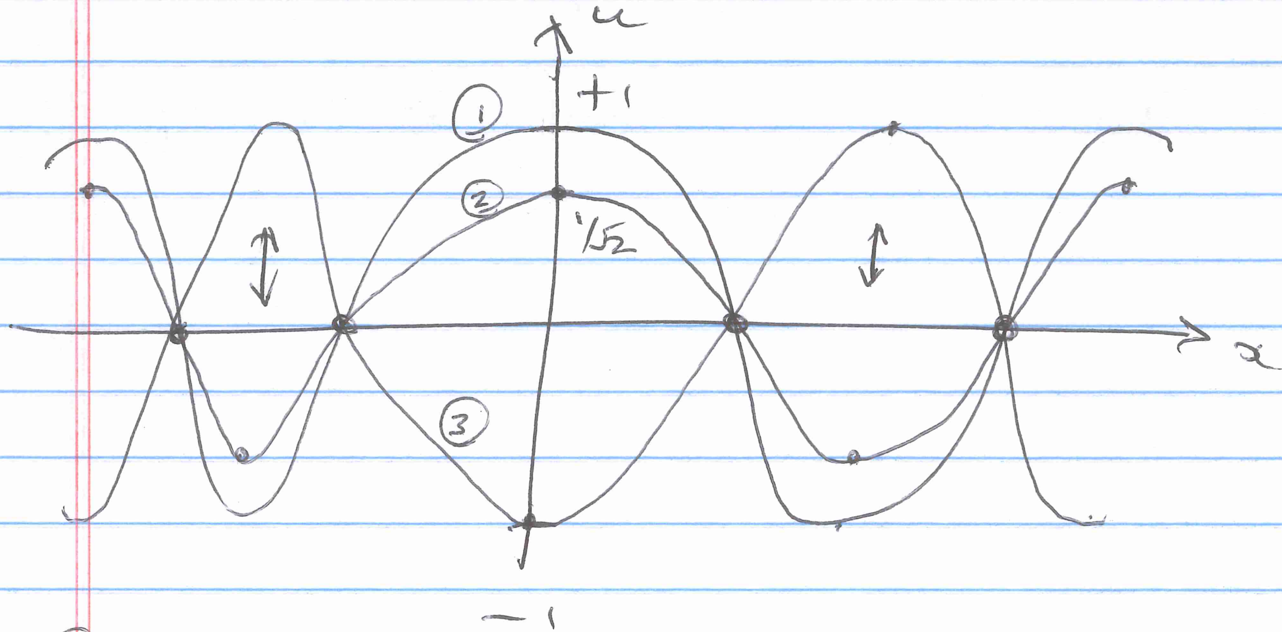
Distributed Initial Displacement
– "Continuous Wave"

$$u(t,x) = \frac{1}{2}\left[\cos(x-ct) + \cos(x+ct)\right]$$

cos + formula $= \cos(ct)\,\cos x$ "STANDING WAVE"

$\cos x$: SHAPE CONSTANT IN TIME

$\cos(ct)$: Amplitude varies over time between ± 1.



(1) $t=0$

(2) $ct=\frac{\pi}{4}$

(3) $ct=\pi$

4 $f(x) = 0, \quad g(x) = \frac{1}{1+x^2}$

ie
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,x) = 0 \\ u_t(0,x) = \frac{1}{1+x^2} \end{cases}$$

STRING AT REST IS PLUCKED TO IMPART AN INITIAL VELOCITY

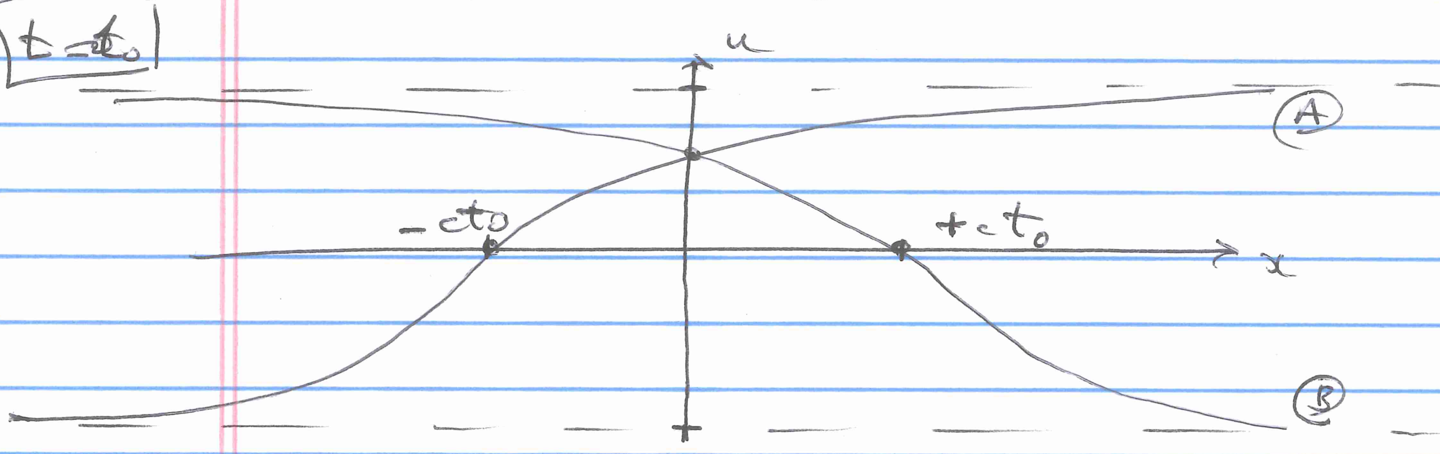
SOLN

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+x^2}\,dx \quad (*)$$

$$= \frac{1}{2c}\left[\arctan(x+ct) - \arctan(x-ct)\right]$$

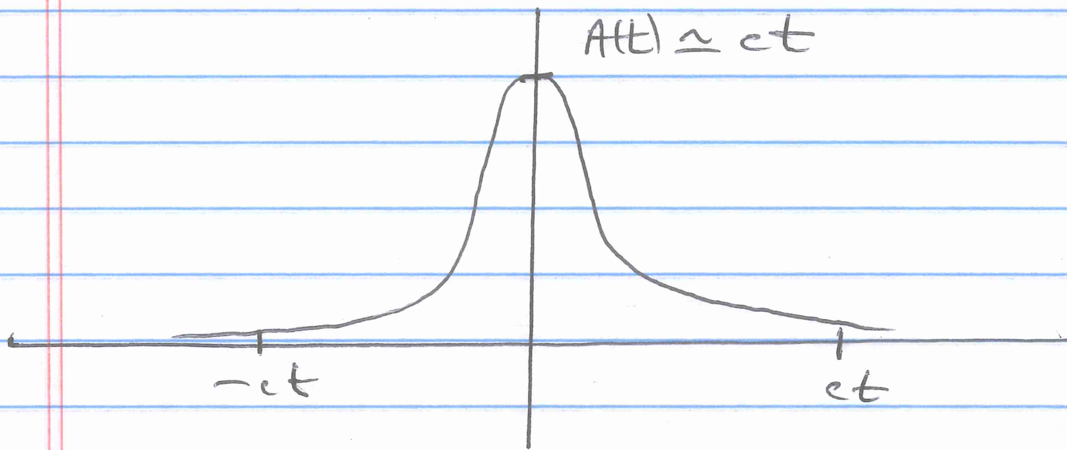
$$= (A) + (B)$$

$\boxed{t = t_0}$



NOTE By $(*)$, $u(t,x) > 0 \quad \forall (t,x)$.

$\boxed{\text{CASE} \quad ct \ll 1}$



$$A(t) = u(0,x) = \frac{1}{2c}\left[\arctan(ct) - \arctan(-ct)\right]$$

$$= \frac{1}{c}\arctan(ct) \qquad (\forall t) \qquad (**)$$

$$= t\,\frac{\arctan(ct) - \arctan(0)}{ct - 0}$$

$$\overset{ct \ll 1}{\simeq} t\,\frac{d}{dt}\arctan(ct)\Big|_{t=0}$$

$$= \frac{ct}{1+(ct)^2} \overset{ct \ll 1}{\simeq} ct$$

ALSO

$$\lim_{t\to\infty} A(t) = \frac{\pi}{2c} \qquad \text{by } (**)$$

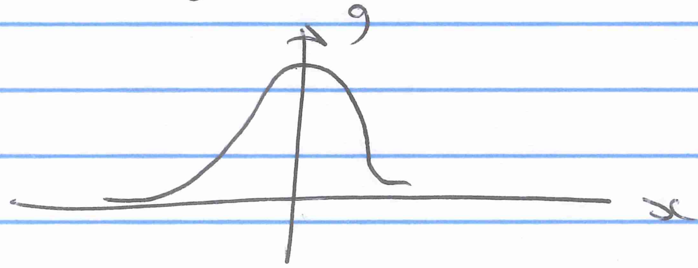
IN GENERAL

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

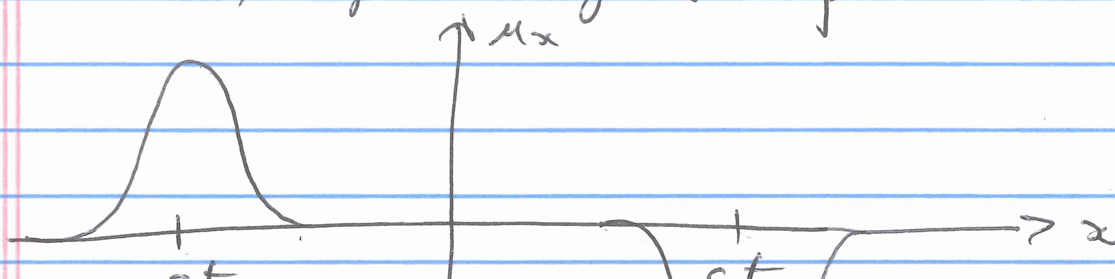
So by FTC

$$u_x(t,x) = \frac{1}{2c}\left[\, g(x+ct) - g(x-ct) \,\right]$$

If $g \geq 0$ and g is a localized pulse, ie



Then for ct large enough (compared to width of g):



So $u(t,x) = \int_{-\infty}^{x} u_x(t,y)\,dy$ gives

See Movie