LECTURE 4 — NONLINEAR TRANSPORT + SHOCKS

INVISCID BURGER'S EQUATION for $u = u(t, x)$ is

$$u_t + u u_x = 0. \qquad (1)$$

NOTES

(a) Eqn (1) is a NONLINEAR transport eqn for which

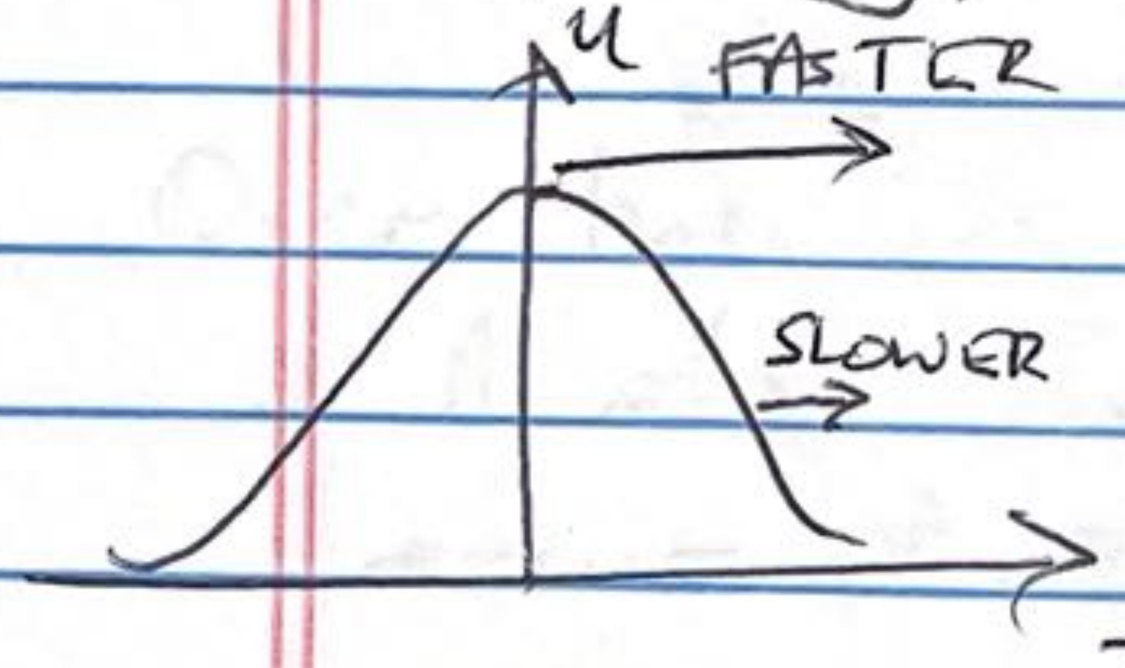
$$c = u$$

VELOCITY = AMPLITUDE

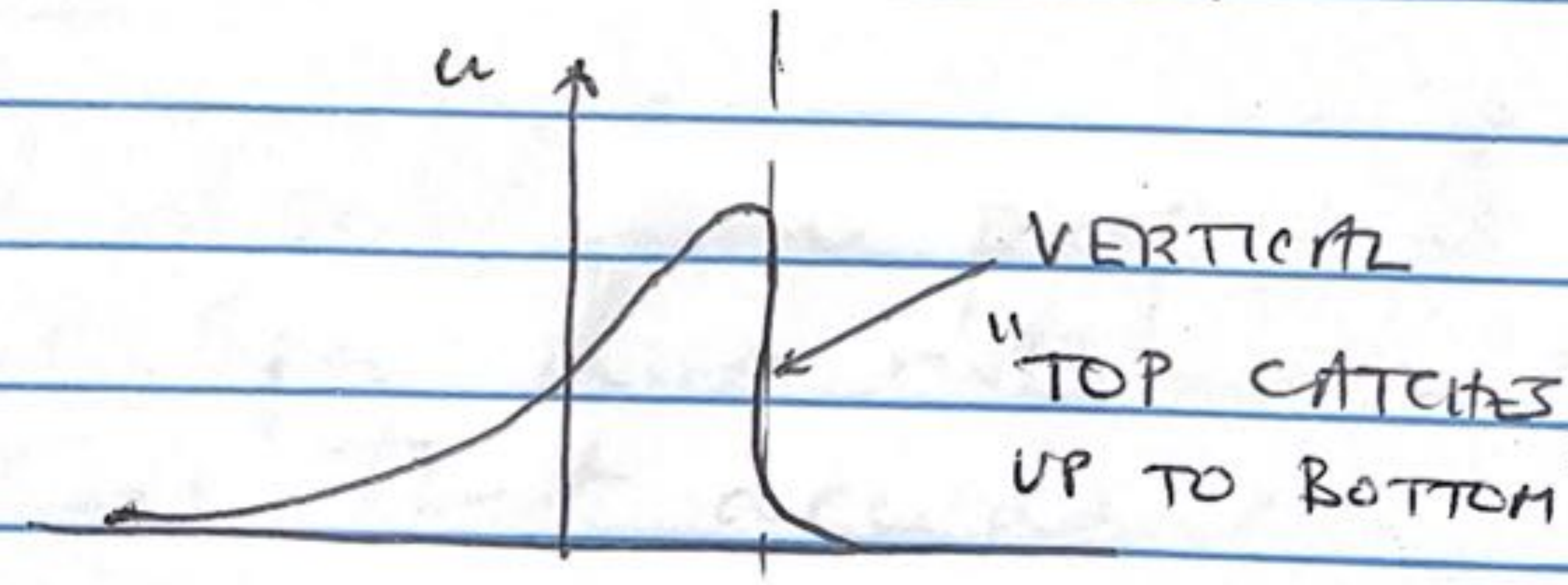
(b) So, if $u > 0$ then wave moves to right
if $u < 0$ then wave moves to left

(c) Also, larger amplitude portions of the wave move faster.

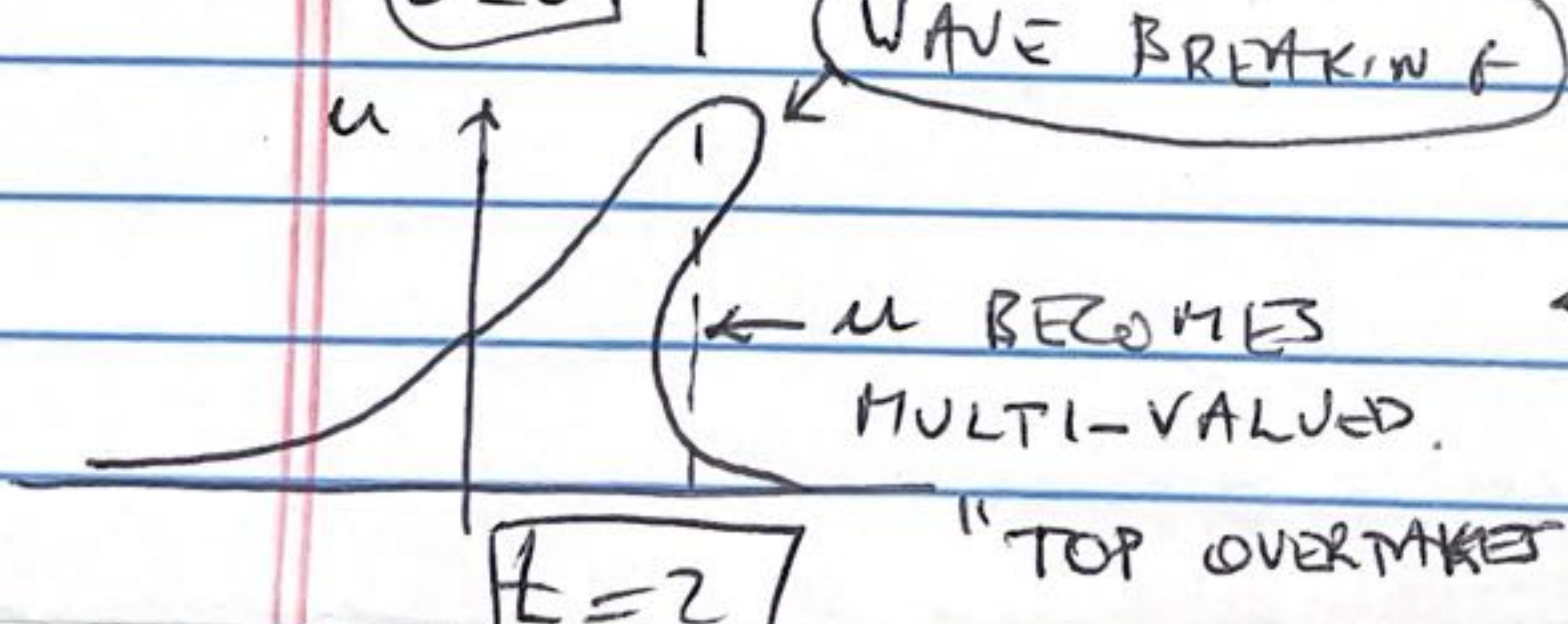
(d) Consequently, the wave can break:



$\boxed{t=0}$



$\boxed{t=1}$



$\boxed{t=2}$ "TOP OVERTAKES BOTTOM"

$\boxed{\text{u IS NO LONGER A CLASSICAL SOLUTION.}}$

ⓔ Generalizations of this PDE model

- Gas dynamics
- Traffic flow
- Flood waves in rivers.

ⓕ In applications such as traffic flow rather than becoming multi-valued, the solution can develop jump discontinuities called <u>SHOCKS</u>.

<u>EG</u> Before you hit a traffic jam, all cars have velocity $u = 40$ mph

When hit jam, all of a sudden all cars stop.

So we get a <u>shock</u>.

Over time the shock itself can propogate
- A slow down at 6pm does not necessarly occur at same place that accident occurred at 5:45 pm

SOLUTION VIA METHOD OF CHARACTERISTICS

Characteristic Curves $x = x(t)$ satisfy ODE

$$\frac{dx}{dt} = u(t, x(t)) \qquad (2)$$

Just as for linear transport we have

CLAIM

If $u = u(t,x)$ solves (1)
Then $h(t) = u(t, x(t))$ is constant
where $x = x(t)$ solves (2)

ie

THE SOLUTION IS CONSTANT ON CHARACTERISTICS

CONUNDRUM

To find u, we need the CCs $x = x(t)$

But to find the CCs we need u!

The way out is given by

KEY IDEA ON CC $x = x(t)$, u is CONSTANT
SO
Since $\frac{dx}{dt} = u$ ON CC, $\frac{dx}{dt}$ = CONST on CC

So slope of CC is const

ie CCs are all STRAIGHT LINES !!

$$\boxed{x(t) = t\underbrace{u}_{\text{CONST}} + k} \quad (3) \quad \text{for const } k$$

SLOPE $= \frac{dx}{dt} = u$ FOR BURGER

The slope of a char line is the value of the sol^n u on that line.

So as for linear transport

CHAR VARIABLE is $\xi = x - tu$

and sol^n is of form

$$\underset{\text{SLOPE}}{u} \underset{=}{=} \underset{\text{VALUE}}{f(\xi)} \quad \text{for some } f^n \ f$$

ie

$$\boxed{u = \underset{\text{GIVEN}}{f}(x - tu)} \quad (4)$$

NOTES (a) f is ~~determined~~ GIVEN by IC

$$u(0, x) = f(x).$$

(b) (4) is an Algebraic Eqn for $u = u(t, x)$.

Probably can't explicitly solve for u.

Even Worse: No gaurantee $\exists!\, u$ for each (t, x).

ie No gaurantee can get a single-valued function u that solves PDE.

SIMPLE EX $f(y) = \alpha y$

$$u = f(x - tu) = \alpha(x - tu) = \alpha x - t\alpha u$$

IN THIS CAN SOLVE:

So $u = u(t, x) = \dfrac{\alpha x}{1 + \alpha t}$

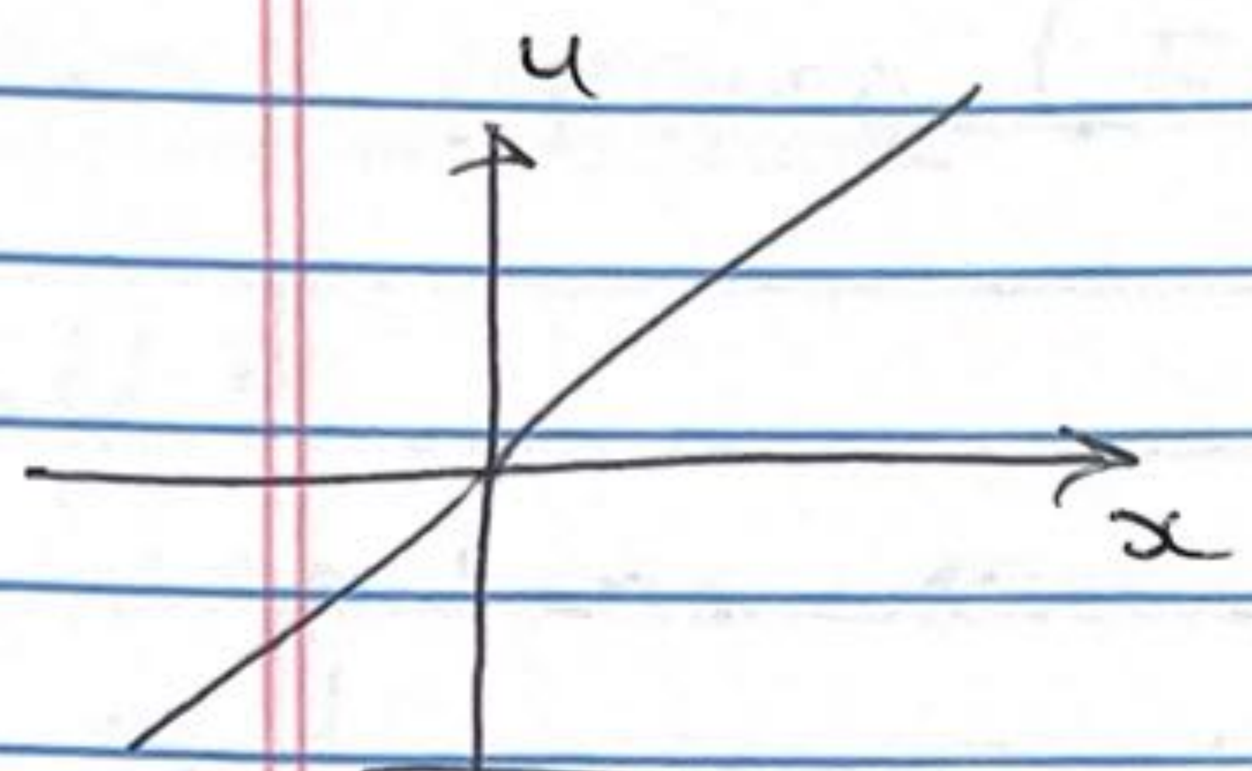
For each fixed t, u is linear function of x with slope

$$m = \frac{\alpha}{1 + \alpha t}$$

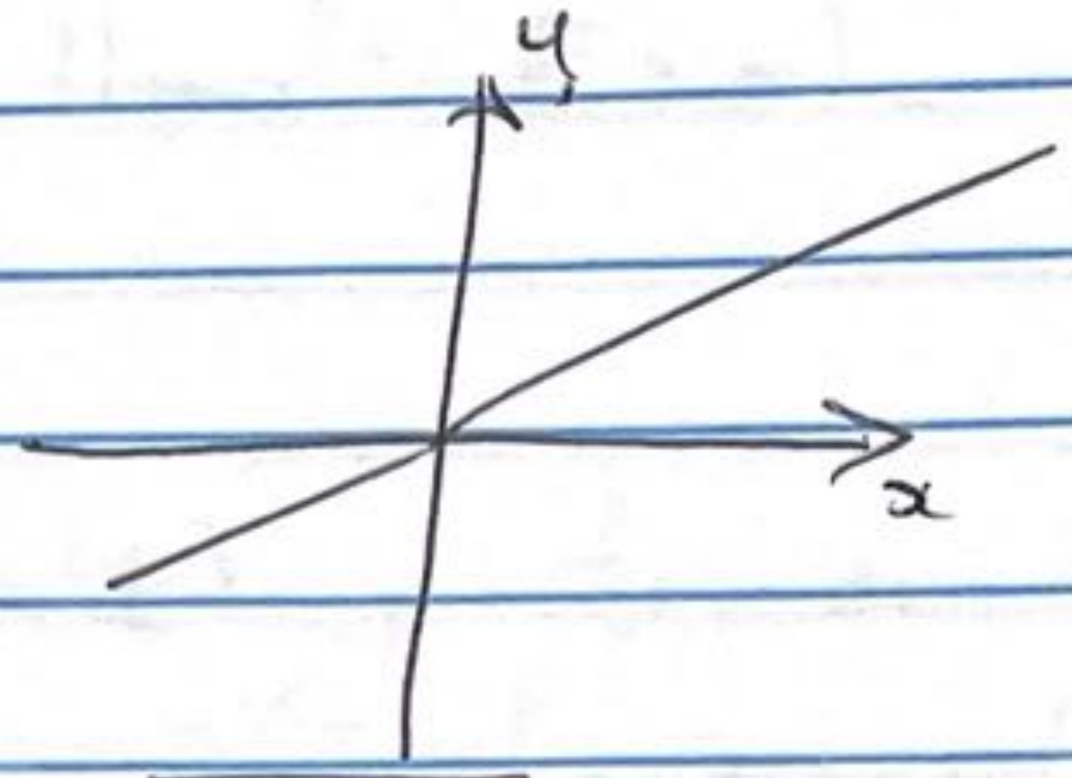
CASE $\alpha > 0$ $(f \uparrow)$

$\forall t > 0$, $1 + \alpha t \neq 0$, so sol^n $\exists\ \forall t > 0$

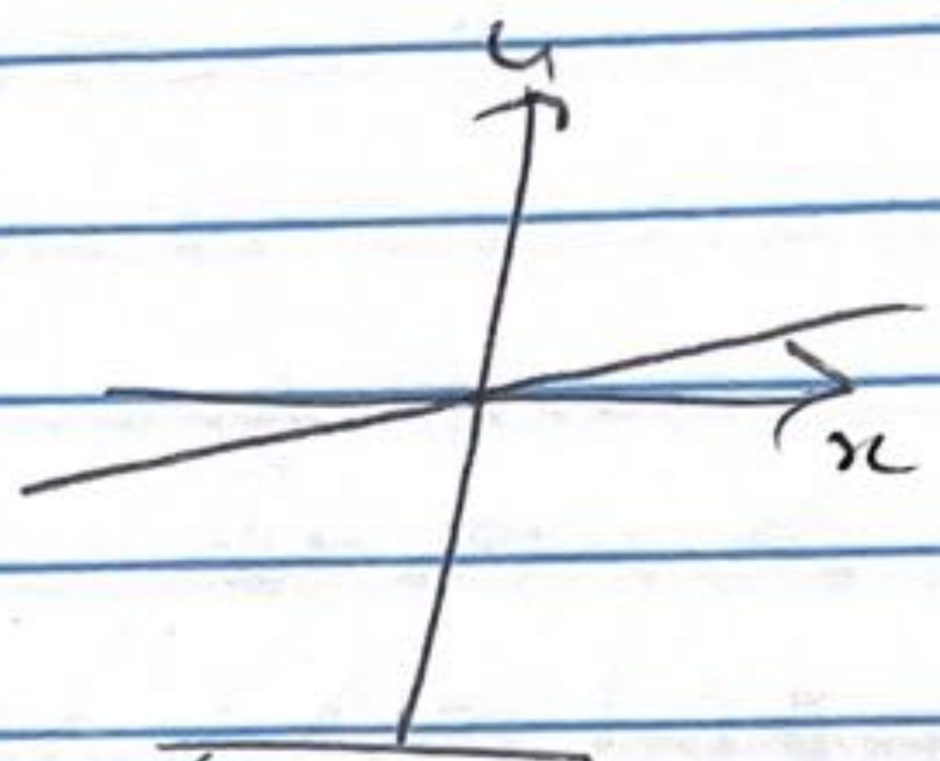
Also $m > 0$ and $m \to 0$ as $t \to +\infty$.



$t=0$



$t=1$



$t=10$

$\boxed{\text{CASE } \alpha<0}$ $(f\downarrow)$

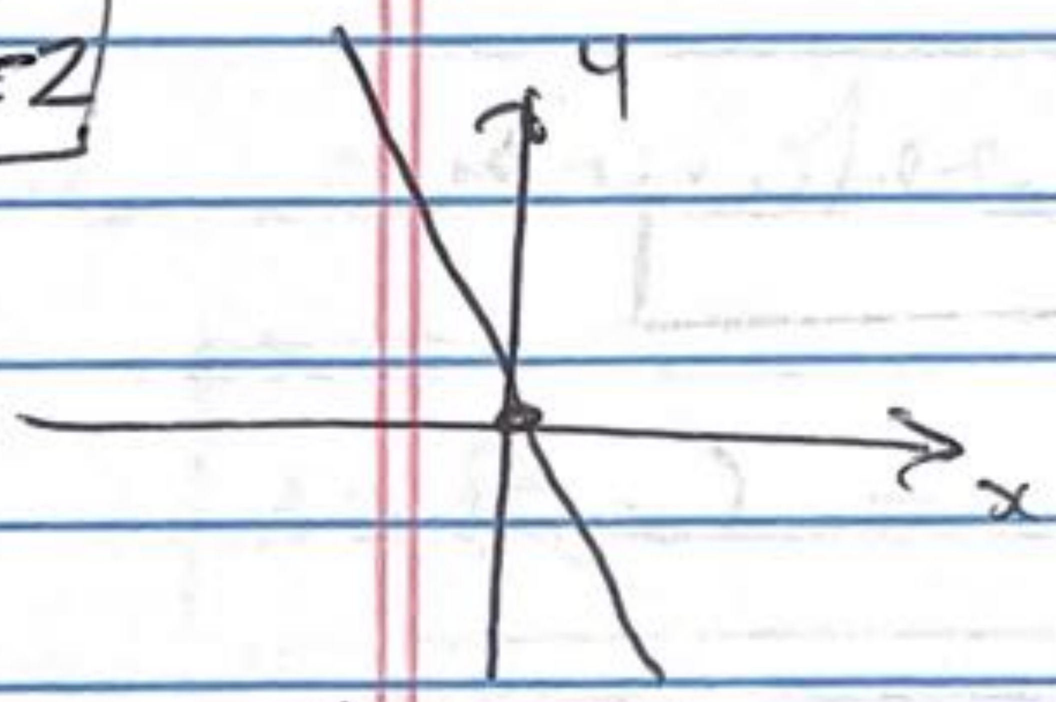
Sol^n only $\exists$ for $1+\alpha t>0$, ie for

$$0<t<t_* := -\frac{1}{\alpha}.$$

Also $m<0$

And as $t\to t_*$, $m\to -\infty$.

$\boxed{\alpha=2}$



$t=0$

$t=\frac{1}{2}-\varepsilon$

GEOMETRIC METHOD (For Harder Exs)

Recall:

Char Line is $x(t) = tu + k$ for some k
and u is constant on each Char Line.

So to solve

$$\begin{cases} u_t + uu_x = 0 & t \geq 0 \\ u(0,x) = f(x) \end{cases}$$

Use fact

Char Line Thru $(0,y)$ has slope $u(0,y) = f(y)$
So eqn is

$$x = t f(y) + y$$

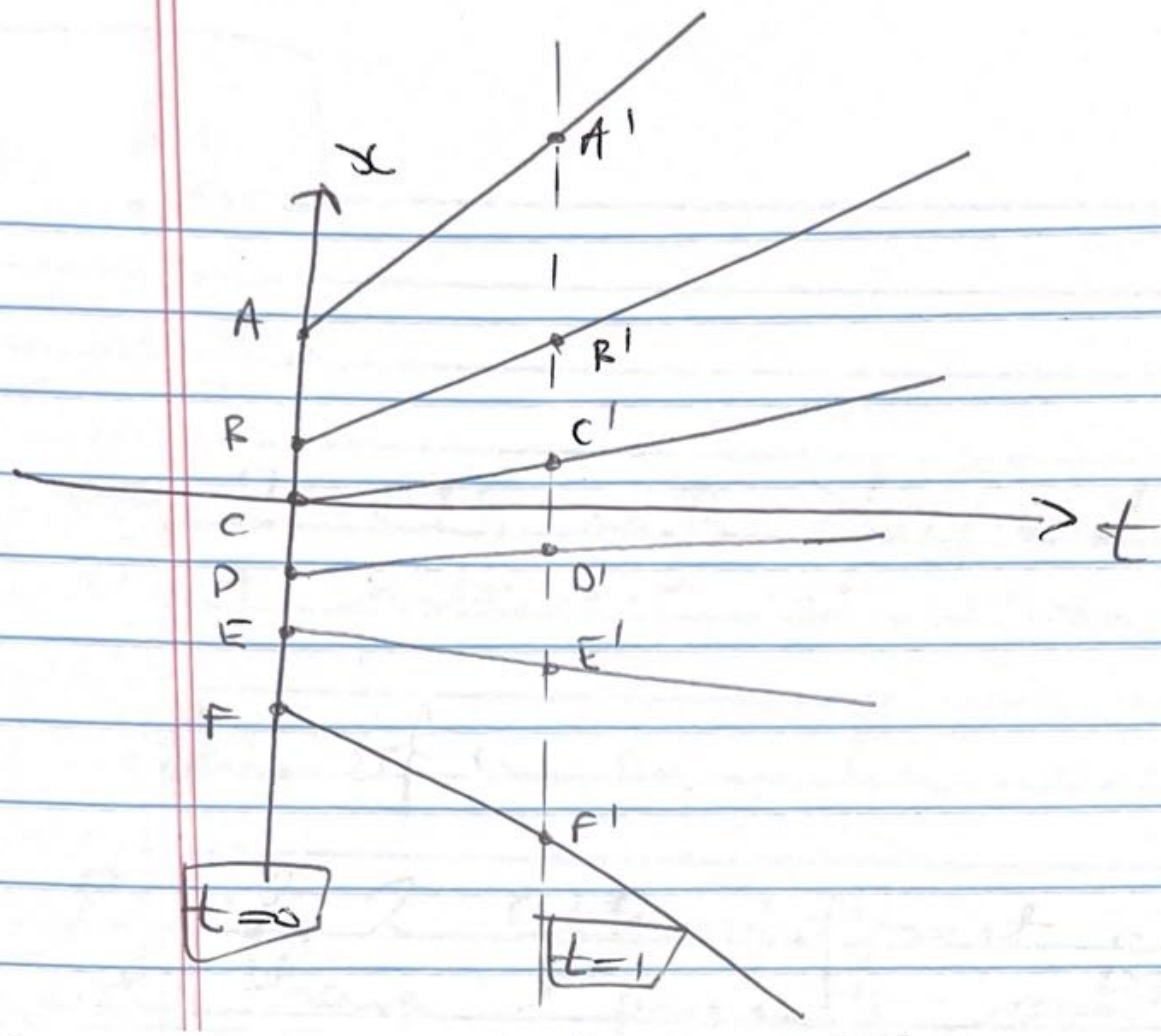
And everywhere on This line value of u is $f(y)$
ie

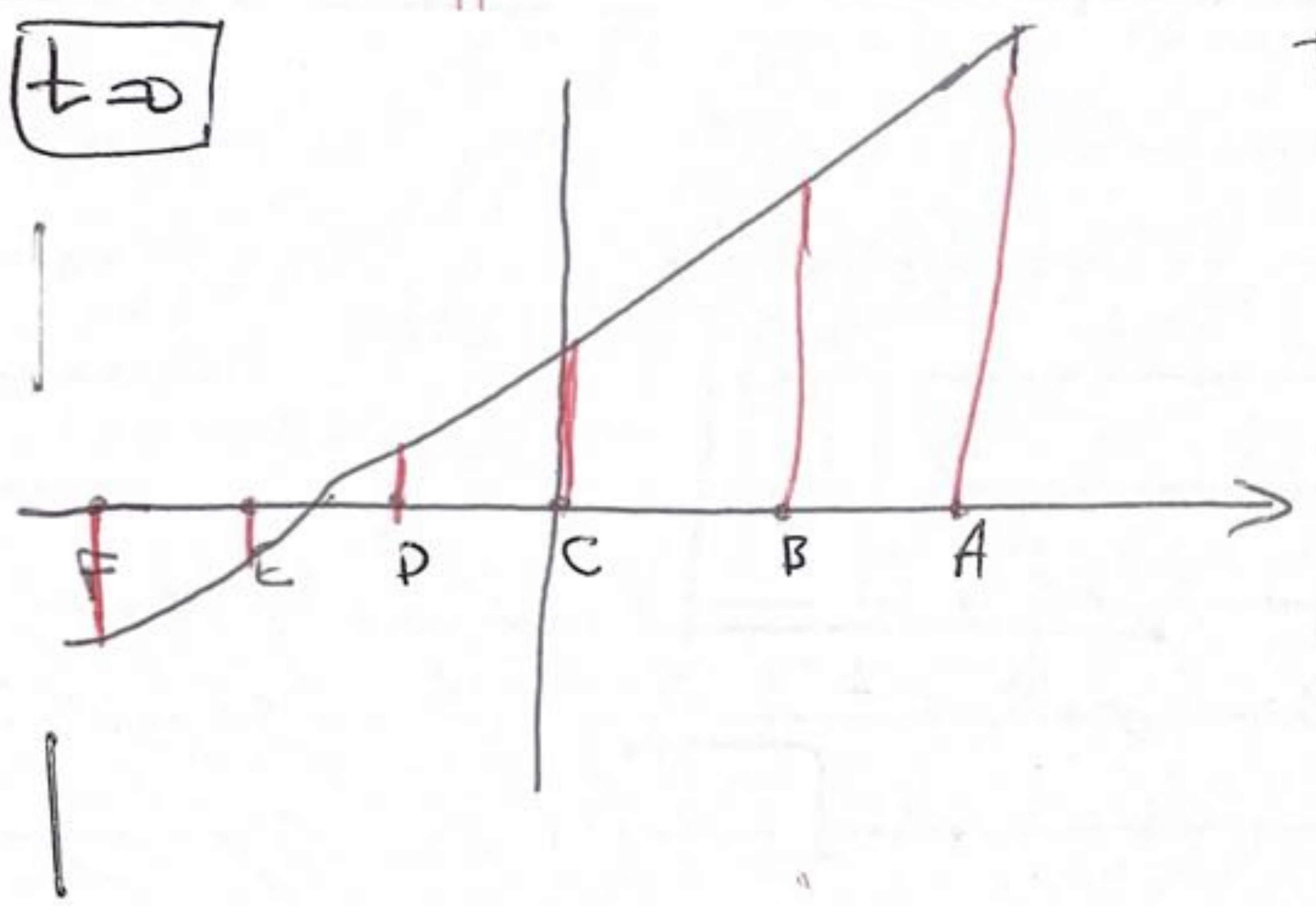
$$\boxed{u(t,x) = f(y) \quad \text{where} \quad x = t f(y) + y}$$

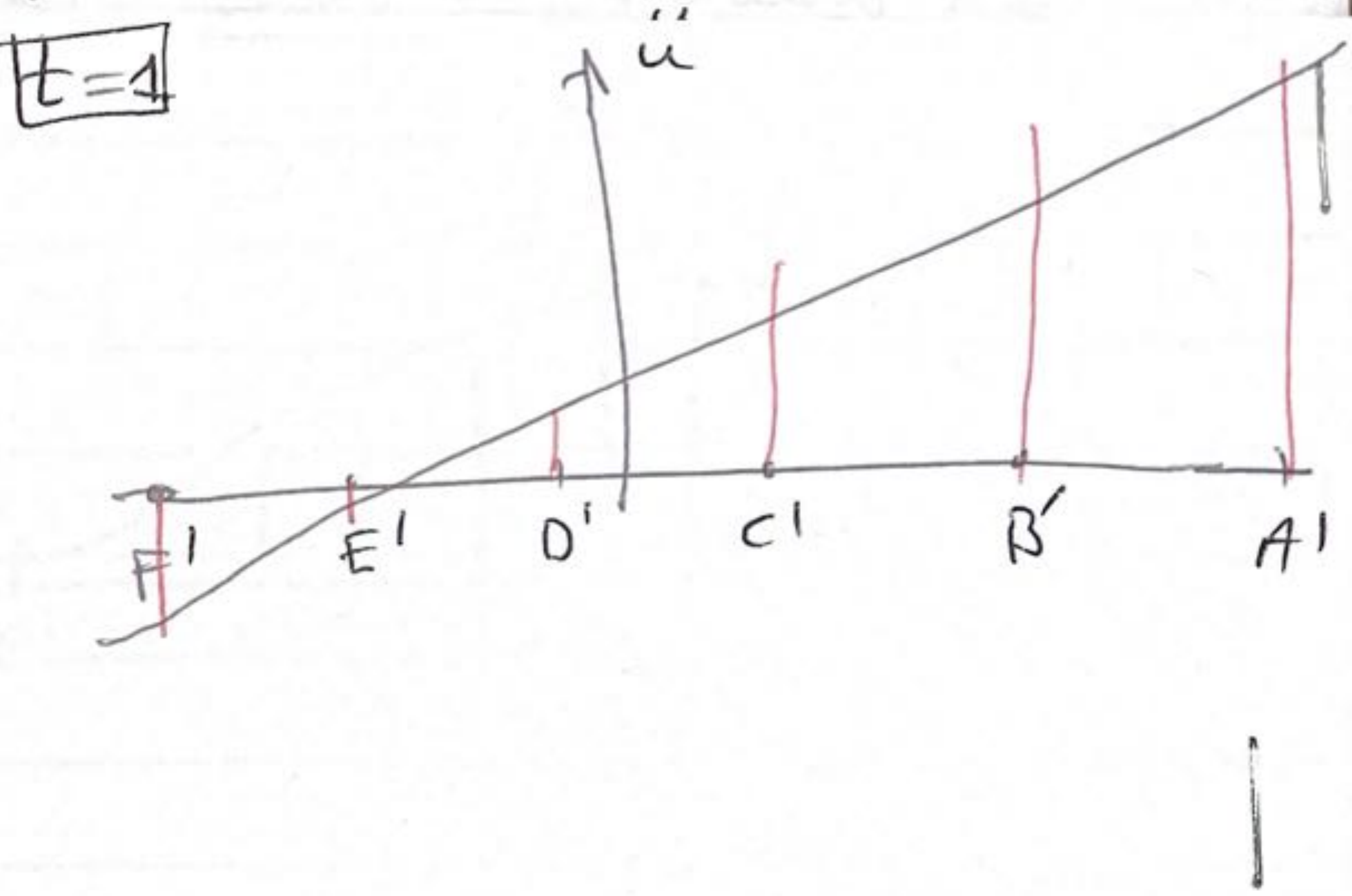
Just like in EX above,
Behaviour of Solution depends on whether
f is $\uparrow$ or $\downarrow$. (If f both $\uparrow$ and $\downarrow$, ~~split problem~~ use similar ideas)

$\boxed{\text{CASE } f \uparrow}$

At $(0,y)$ on x-axis, $m = f(y)$
So as $y \uparrow$, Slope of CL $\uparrow$.







As $t \uparrow$, wave spreads out.

Say have a <u>RAREFACTION WAVE</u>

<u>REASON</u> <u>Rarefaction</u> = ↓ in Density.
The vertical sticks get more spread out, less dense, as $t \uparrow$.

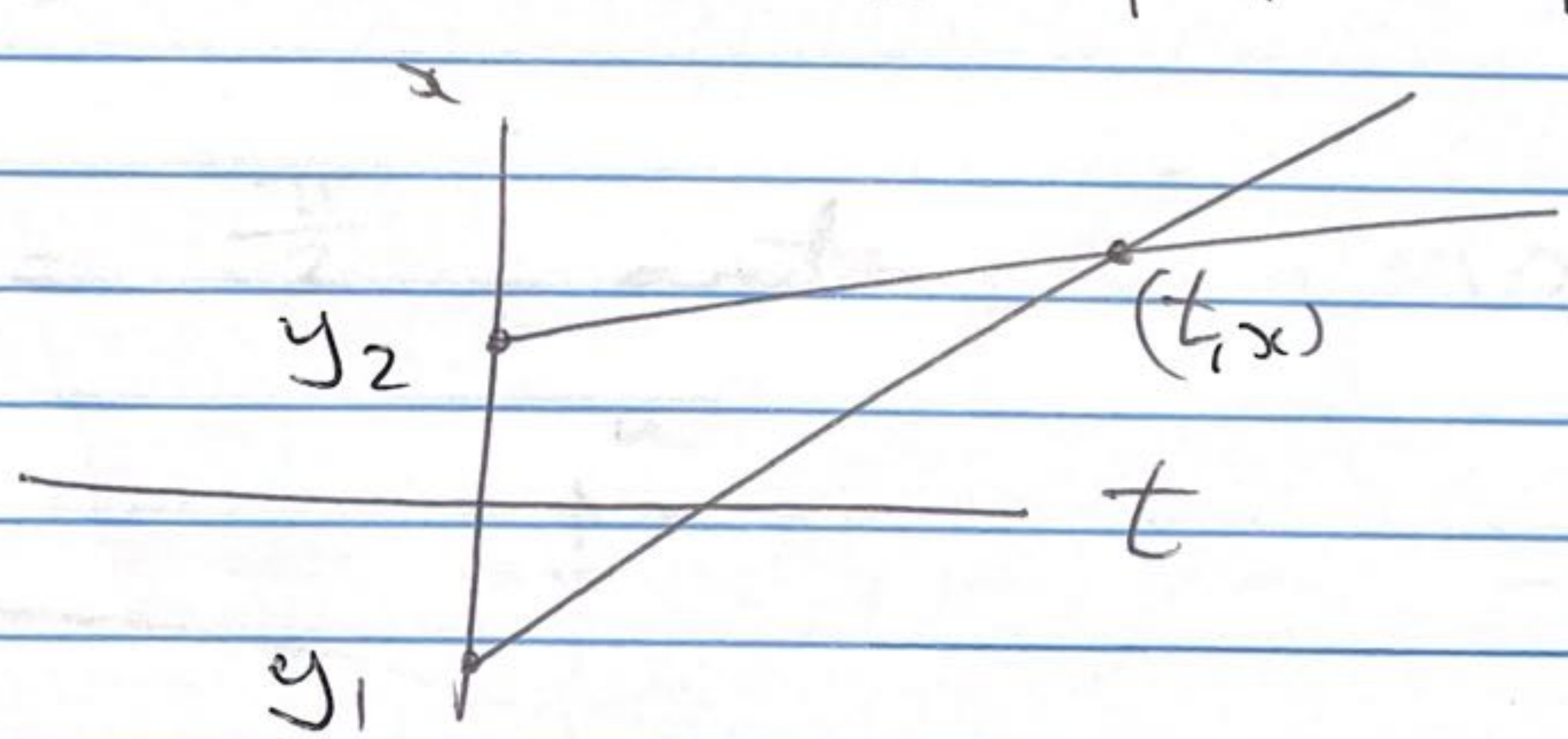
cf <u>Compression</u> = ↑ in Density

$\boxed{\text{CASE } f\downarrow}$

CONUNDRUM

1. Any 2 CL's with different slopes must intersect at some point (t,x)
2. Value of u on a CL = Slope of CL
3. So if 2 CL's intersect at (t,x) we have 2 possible ~~set~~ values for $u(t,x)$.
4. If $f\downarrow$, Then any 2 CL's intersect at a pt (t,x) where $t>0$

$$y_1 < y_2 \implies \text{SLOPE}_1 = f(y_1) > f(y_2) = \text{SLOPE}_2$$



GOOD NEWS: If f' does not get too large negative Then $\exists$ CRITICAL TIME $t_* > 0$ ~~so~~ that the intersectⁿ of any 2 CL's occurs at (t,x) with $\boxed{t \geq t_*}$ ~~$0 \leq t \leq t_*$~~

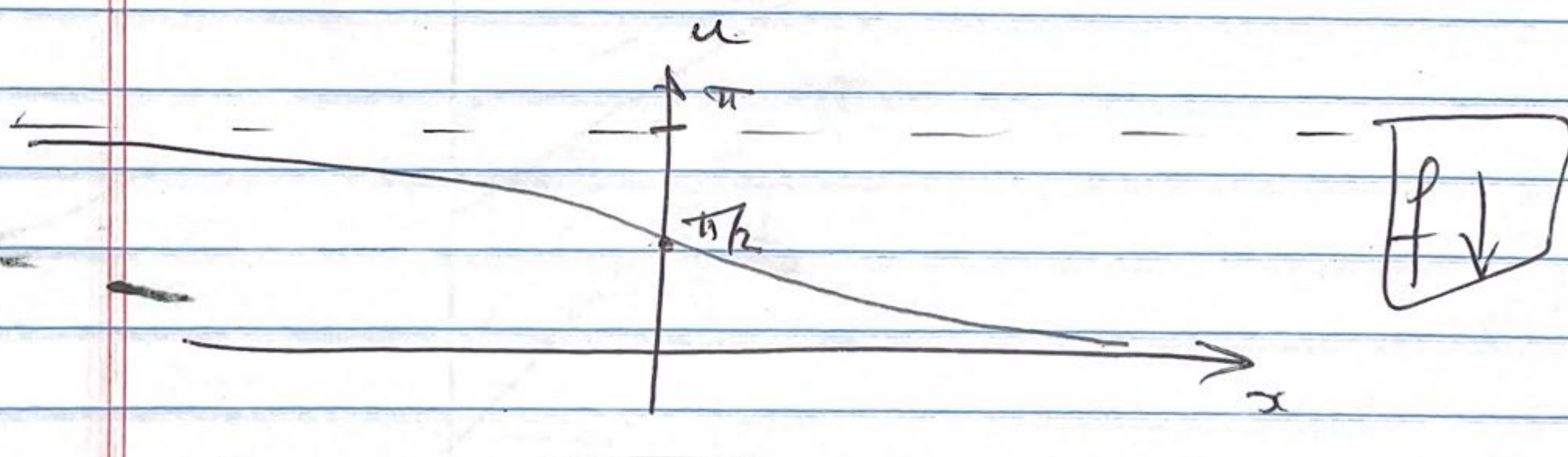
So we have a single-valued solⁿ $u = u(t, x)$

$\forall (t, x)$ with $\boxed{0 \leq t < t_*}$

$\boxed{\text{3 OPTIONS FOR } t \geq t_*}$

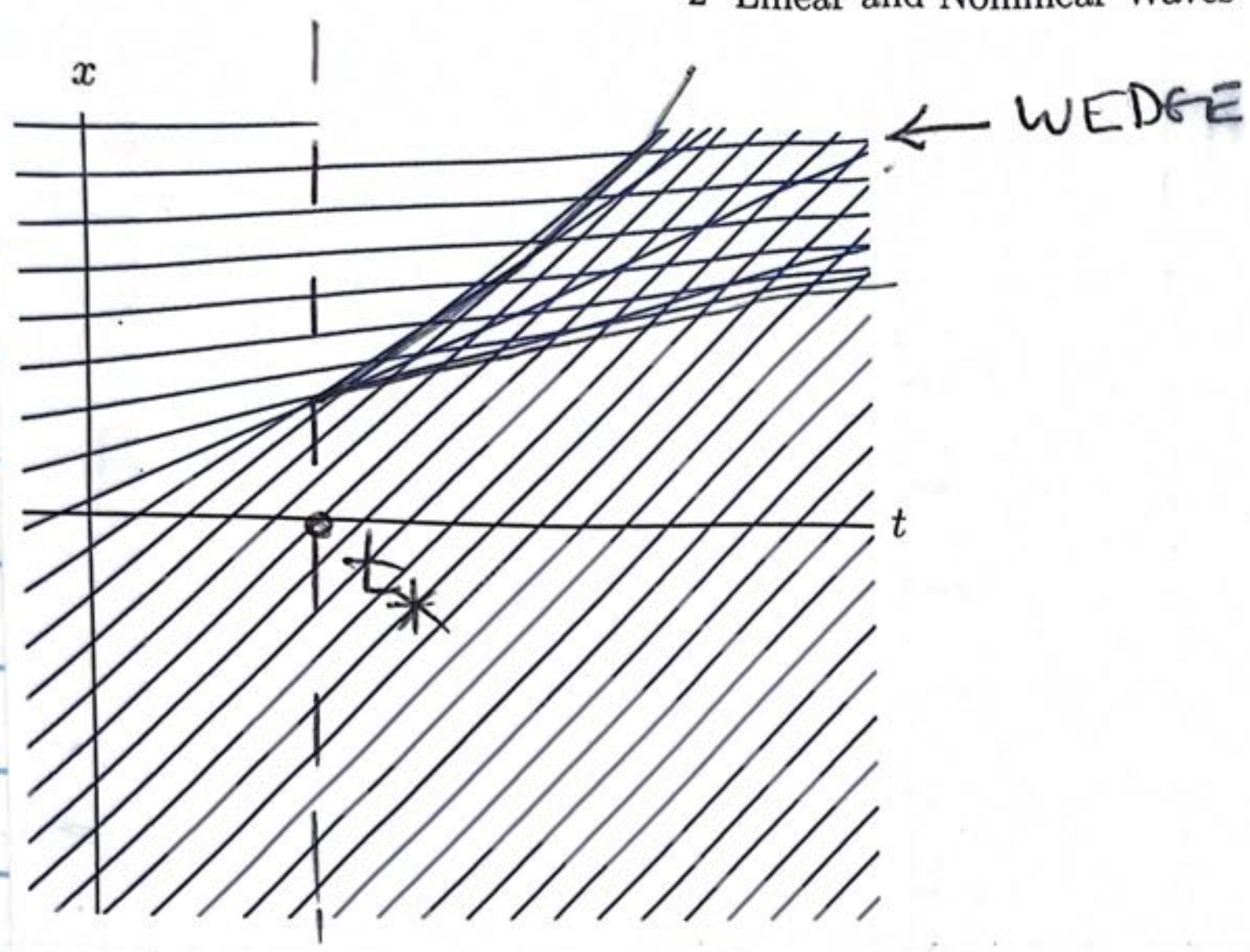
1. SOLN DNE
2. SOLN IS MULTIVALUED FUNCTION
   - Like Breaking Wave
3. We use some other physical criterion to pick one of the solutions.
   - Approach when modeling gas dynamics, traffic flow, etc.
   - Beyond scope of our course.

EX $f(x) = \frac{\pi}{2} - \arctan x = u(0, x)$

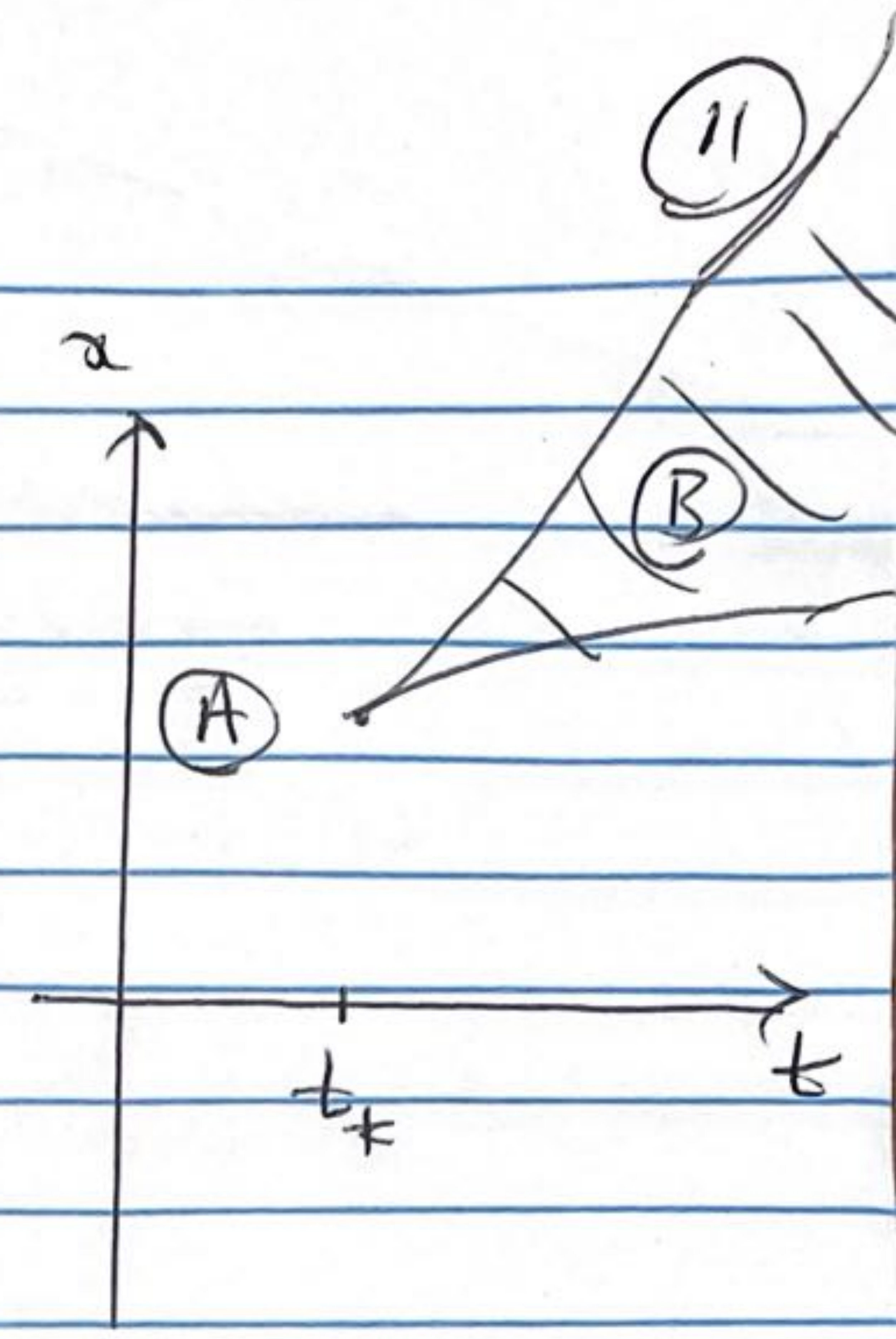


$\boxed{t = 0}$

2 Linear and Nonlinear Waves



Characteristics lines for $u(0,x) = \frac{1}{2}\pi - \tan^{-1} x$.



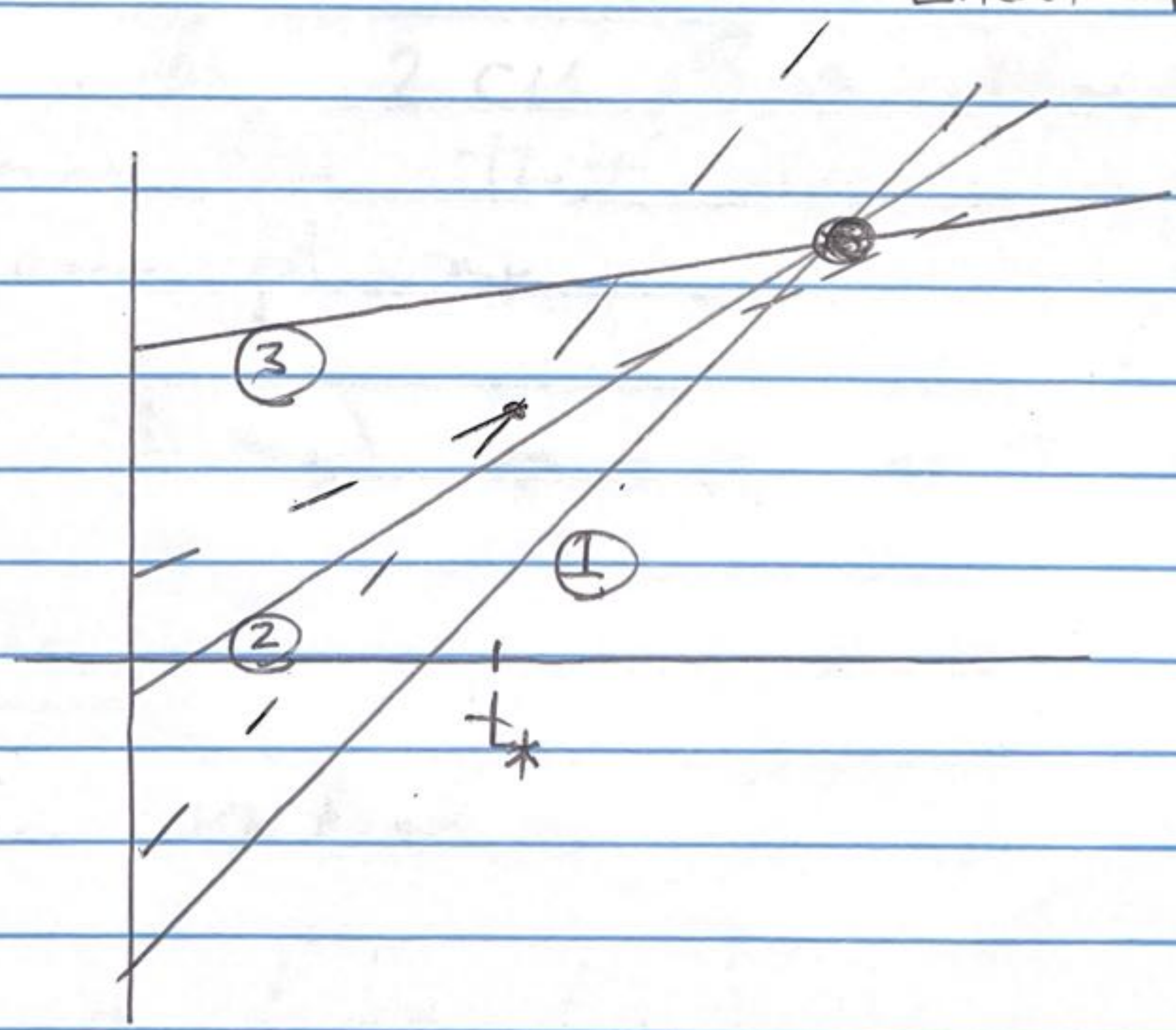
A : 1 CL THRU EACH PT

B : 3 CL's THRU EACH PT

AS $x \to -\infty$, SLOPE $\to \pi$

AS $x \to 0$, SLOPE $\to 0$

ON BORDER : 2 CL's THRU EACH PT.



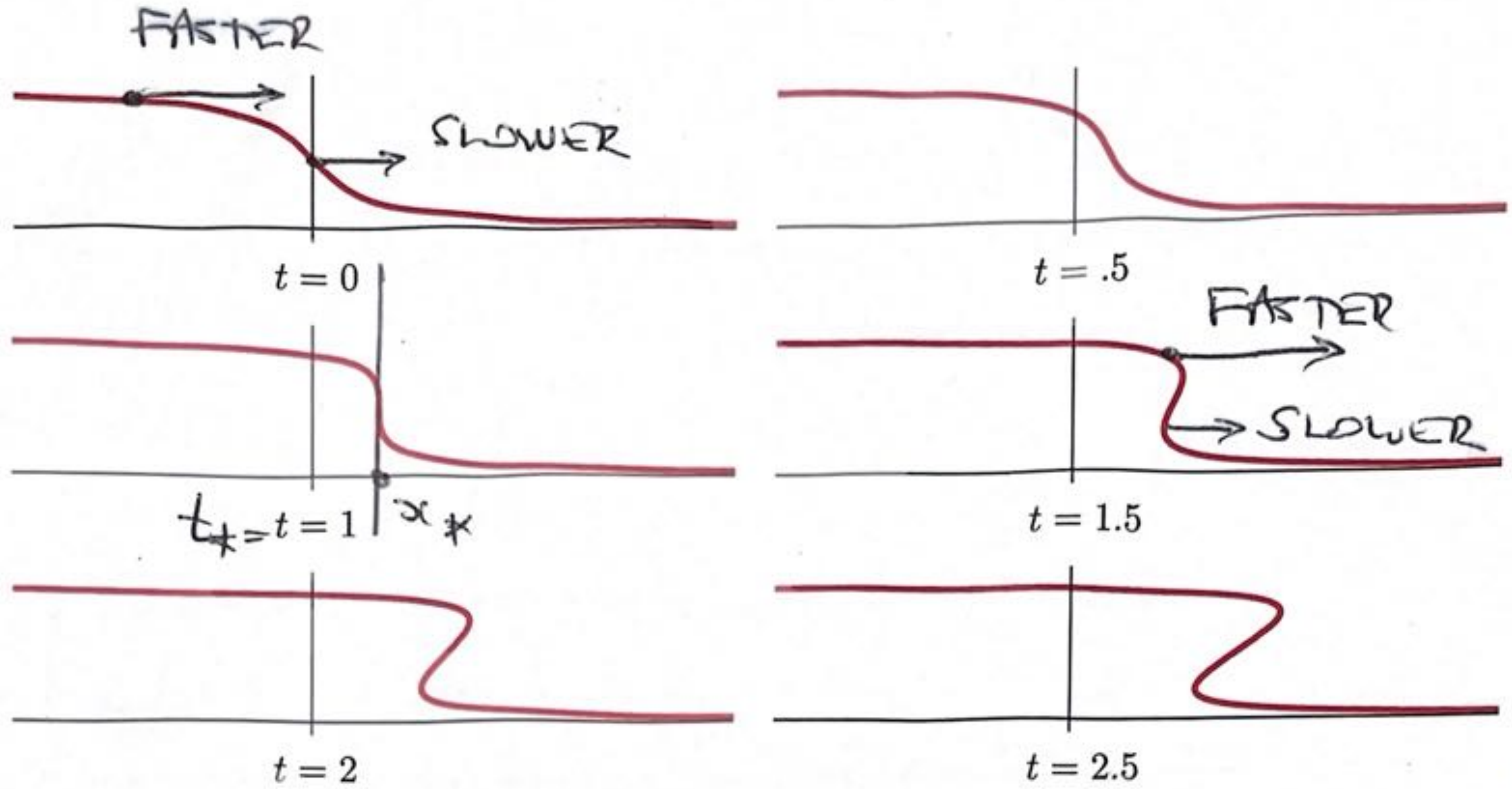


Figure 2.15. Multiply valued compression wave.

Since SPEED = HEIGHT, and $f \downarrow$ PTS further to left move faster than those to right and wave breaks.

The time t_* when 2 Cl's first intersect occurs at same time that slope of sol^n is vertical at some pt x_*

ie $\quad \frac{\partial u}{\partial x}(t, x_*) \longrightarrow -\infty \quad$ as $t \to t_*$.

$\boxed{\text{FORMULA FOR } t_*}$

We know u satisfies

$$u = f(x - tu)$$

So $u_x = f'(x - tu)\,(1 - t\,u_x)$

Solving for u_x :

$$u_x = \frac{f'(\xi)}{1 + t f'(\xi)} \qquad \xi = x - tu$$

So $u_x \to -\infty$ as $t \to \dfrac{-1}{f'(\xi)}$.

The CRITICAL TIME t_* is therefore ~~smallest~~ earliest such time,

$$t_* = \min \left\{ \frac{-1}{f'(x)} \;\middle|\; f'(x) < 0 \right\}$$

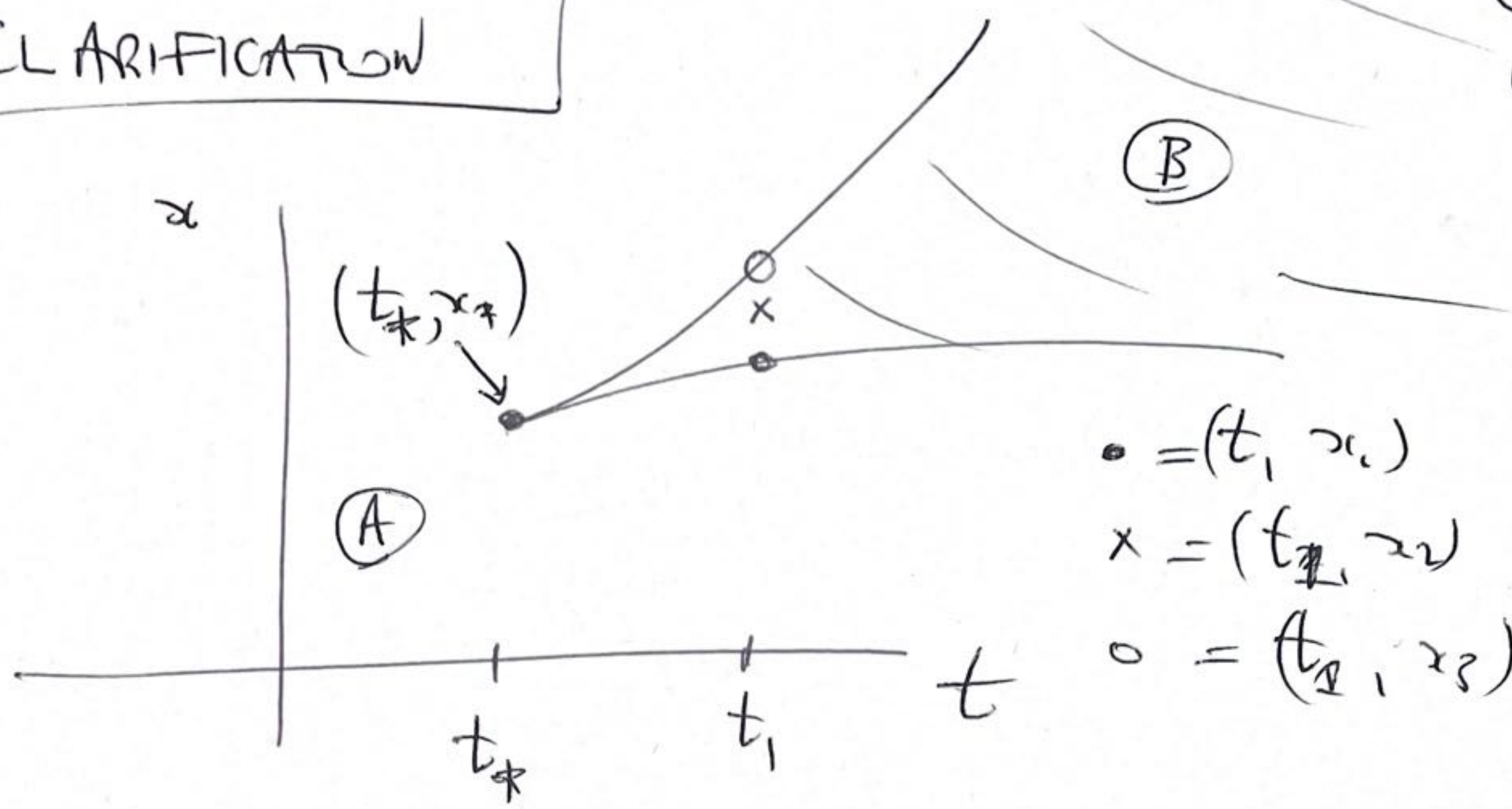
EX

$$f(x) = \pi/2 - \arctan x$$

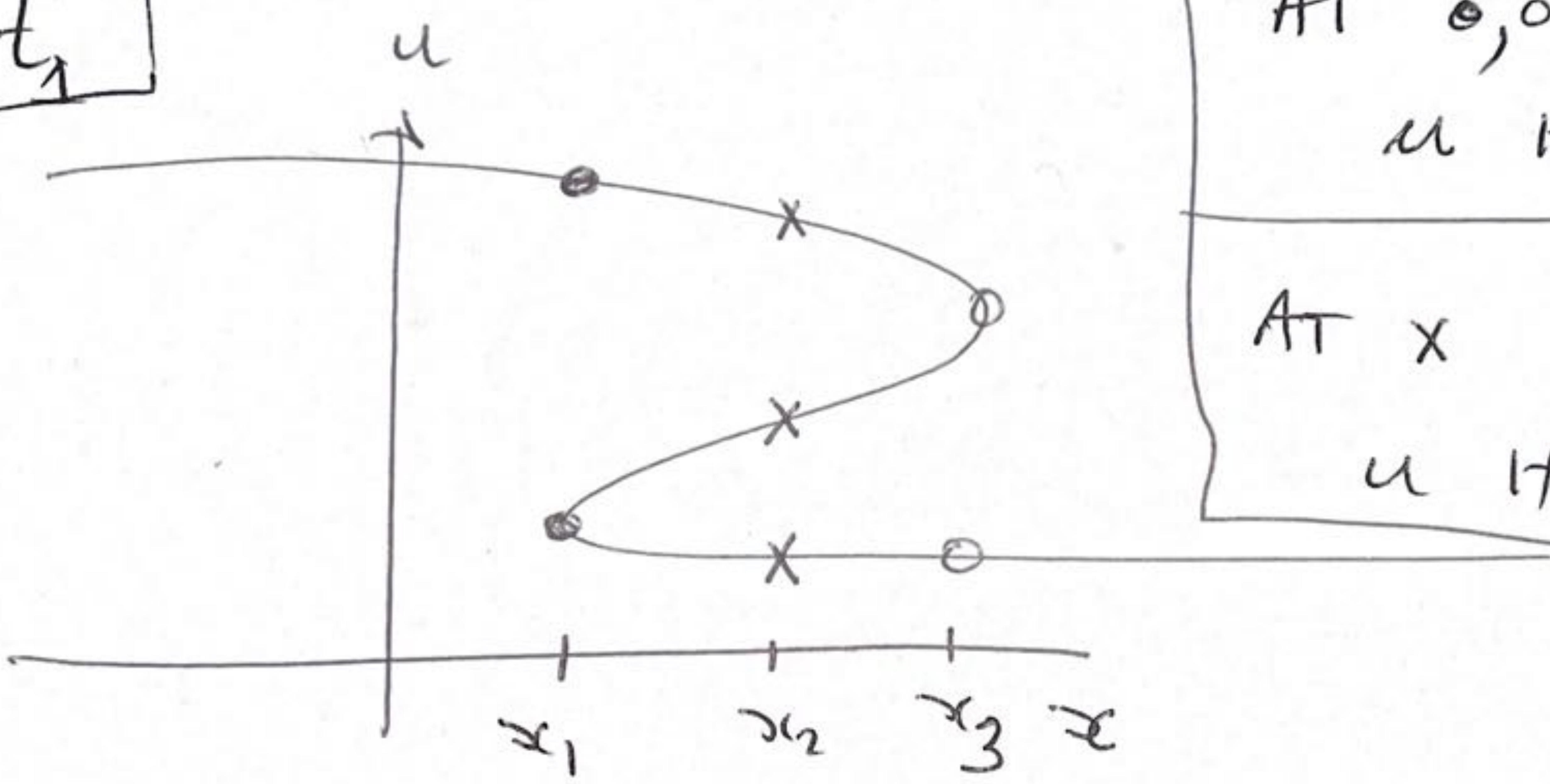
$$f'(x) = \frac{-1}{1+x^2}$$

$$t_* = \min \left(\frac{-1}{\frac{-1}{1+x^2}} \right) = \min\,(1+x^2) = 1$$

CLARIFICATION



$\boxed{t = t_1}$



AT $\bullet, \circ$ 2 CL's CROSS
u HAS 2 VALUES

AT $\times$ 3 CL's CROSS
u HAS 3 VALUES

$\boxed{t = t_*}$

AT (t_*, x_*) HAVE INSTANTANEOUS VERTICAL SLOPE ON GRAPH OF u. BUT THIS DOES NOT MEAN $\exists$ ∞ # CL's THRU (t_*, x_*). AND IT DOES NOT MEAN u TAKE AN ∞ # VALUES (VERTICAL GRAPH) AT (t_*, x_*).

JUST LIKE $g(x) = -x^{1/3}$ HAS $g'(x) \to -\infty$ AS $x \to 0$ BUT g IS ALWAYS SINGLE VALUED FUNCTION, SO WE CAN HAVE $v(x) = u(t_*, x)$ WITH $v'(x) \to -\infty$ AS $x \to x_*$ BUT v SINGLE VALUED