LECTURE 3. LINEAR AND NONLINEAR WAVES [CONT'D]

EX (EX 2.5)

$$\frac{\partial u}{\partial t} + (x^2-1)\frac{\partial u}{\partial x} = 0 \qquad (1)$$

ODE for Characteristic Curves

$$\frac{dx}{dt} = x^2 - 1 \qquad (2)$$

NEW ISSUE: $c(x) = x^2 - 1$ changes sign as x changes.

QUALITATIVE ANALYSIS

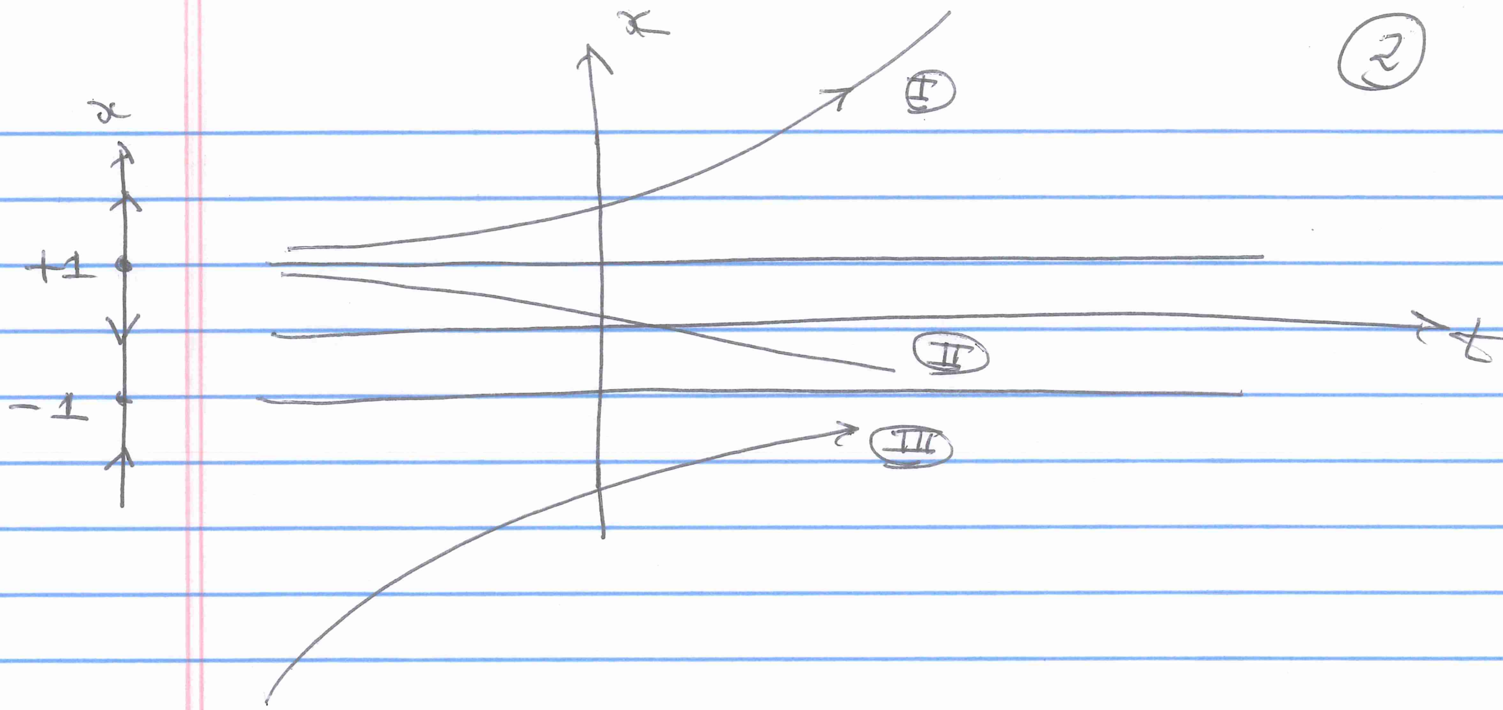
- Equilibrium Sol^{ns} $x(t) = \text{CONST}$ have

$$0 = \frac{dx}{dt} = x^2 - 1 = (x-1)(x+1)$$

So $x(t) = +1$ or $x(t) = -1$ ✓

- Where $|x| > 1$, $x^2 - 1 > 0$ so $\frac{dx}{dt} > 0$
  So $x = x(t)$ is $\uparrow$.

- Where $|x| < 1$, $x = x(t)$ is $\downarrow$.



FORMULAE

$$\frac{dx}{dt} = x^2 - 1$$

$$\int \frac{dx}{x^2-1} = \int dt$$

$$\beta(x) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = t + k \qquad (3)$$

$$\left|\frac{x-1}{x+1}\right| = e^{2(t+k)} \qquad (4)$$

$\boxed{\text{CASE I, III}}$ $\quad |x(t)| > 1 \quad \forall t$

IF $x > 1$ then $\dfrac{x-1}{x+1} > 0$

IF $x < -1$ then $x - 1 < x + 1 < 0$ so $\dfrac{x-1}{x+1} > 0$

So $$\frac{x-1}{x+1} = e^{2(t+k)}$$

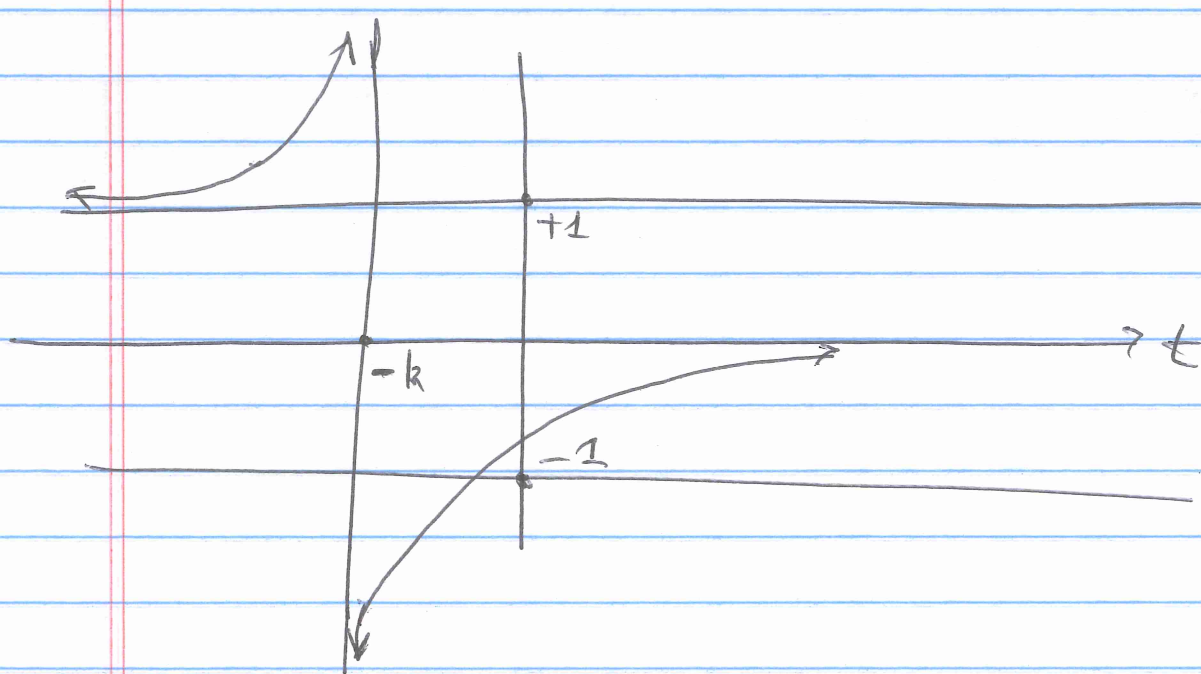
Solve for x to get

$$x(t) = \frac{1 + e^{2(t+k)}}{1 - e^{2(t+k)}}$$

Soln DNE at $t = -k$.

As $t \to -k^-$, $x \to +\infty$
As $t \to -k^+$, $x \to -\infty$.

As $t \to -\infty$, $x \to +1$
As $t \to +\infty$, $x \to -1$



~~CASE $k \geq 0$~~

As $k \uparrow$, char curve translates left.
as formula is of form
$$\beta(x) = t + k$$

$\boxed{\text{CASE II}}$ $|x(t)| < 1 \quad \forall t$

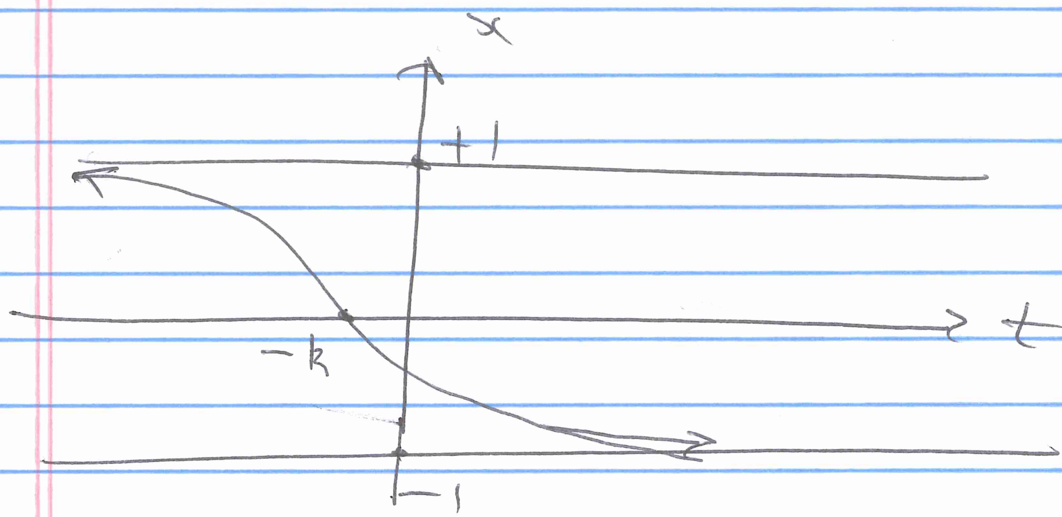
So $\dfrac{x-1}{x+1} < 0$

So
$$x(t) = \frac{1 - e^{2(t+k)}}{1 + e^{2(t+k)}} = \frac{e^{-2(t+k)} - 1}{e^{-2(t+k)} + 1}$$

So $\forall k$ SOLN $\exists$ $\forall t$

As $t \to -\infty$, $x \to +1$
As $t \to +\infty$, $x \to -1$

When $x = 0$, $t = -k$



By CTY all CC cross x-axis.

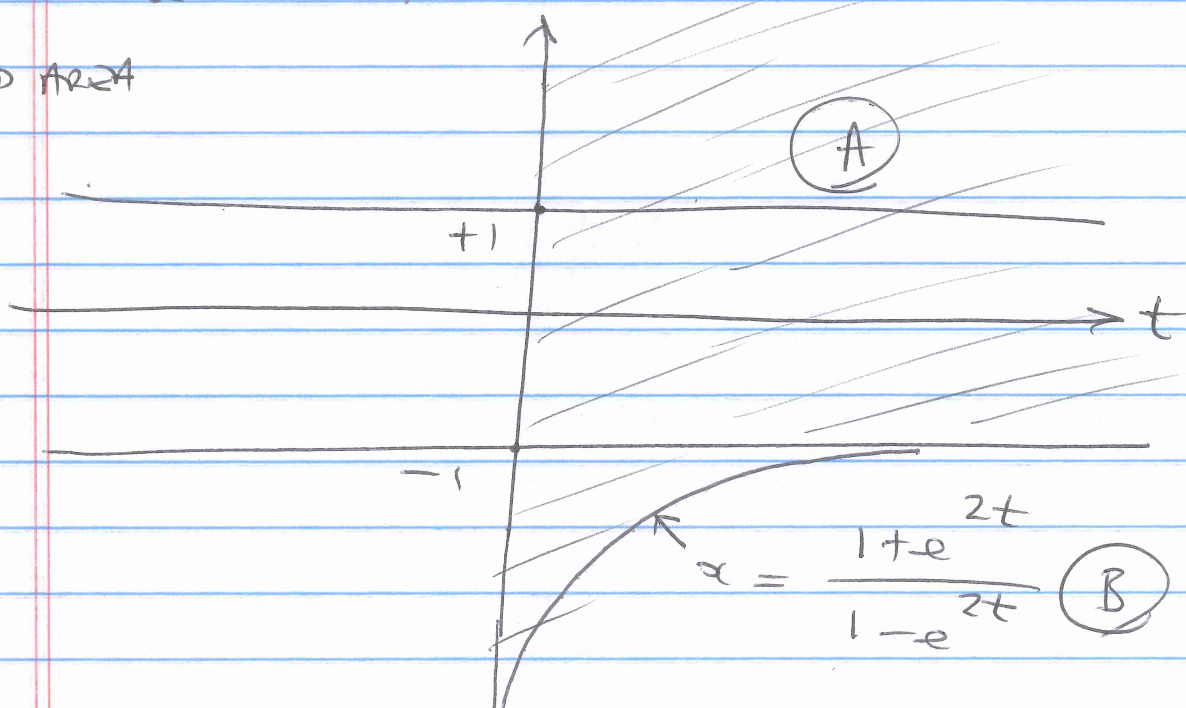
As $k \uparrow$, char curve translates left

NOW SOLVE PDE IVP

$$\begin{cases} u_t + (x^2-1)u_x = 0 & t \geq 0 \\ u(0,x) = e^{-x^2}. \end{cases}$$

The solution is determined at all (t,x) with $t \geq 0$ and (x,t) on a CC that crosses x-axis.

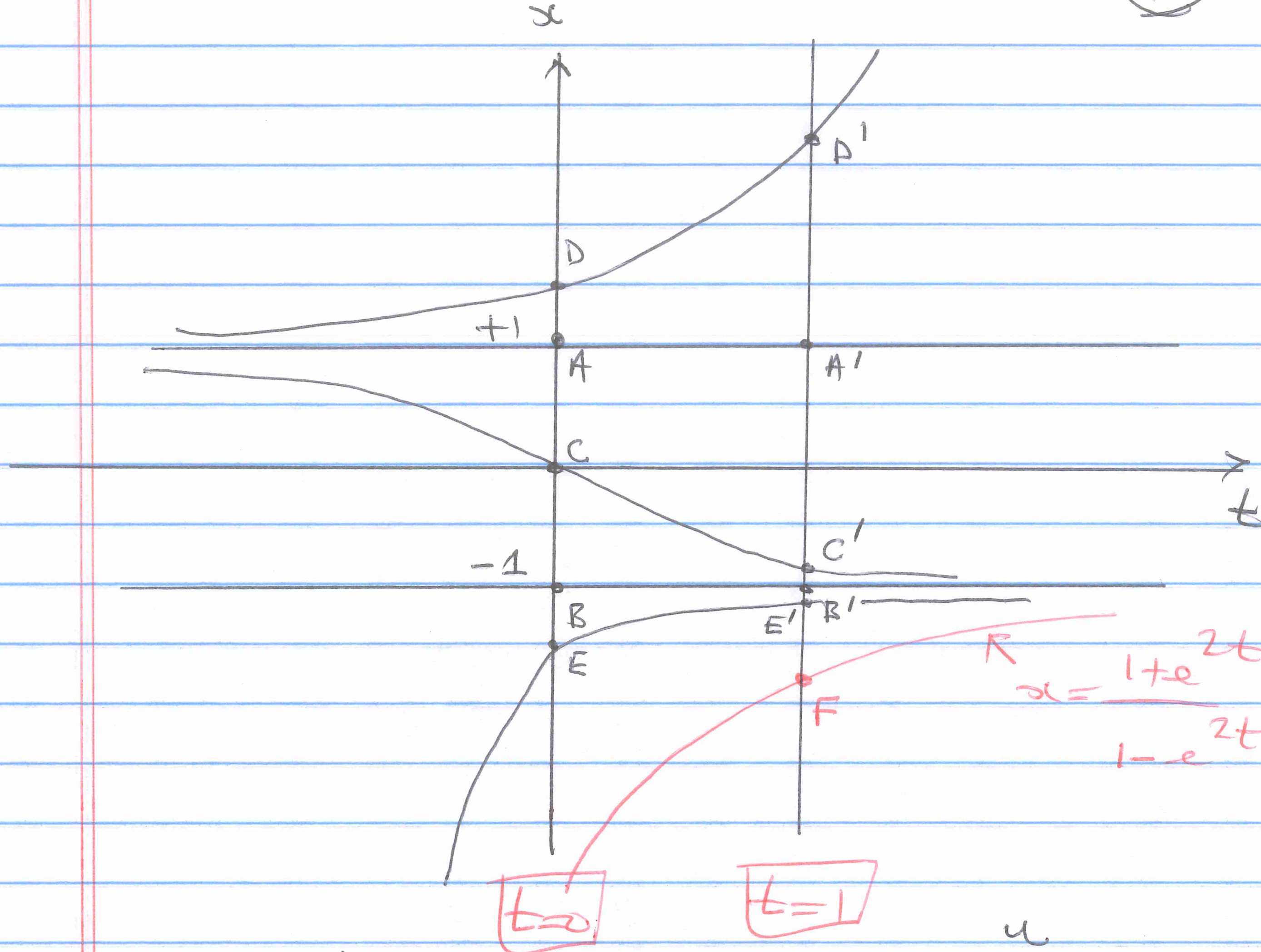
ie SHADED AREA

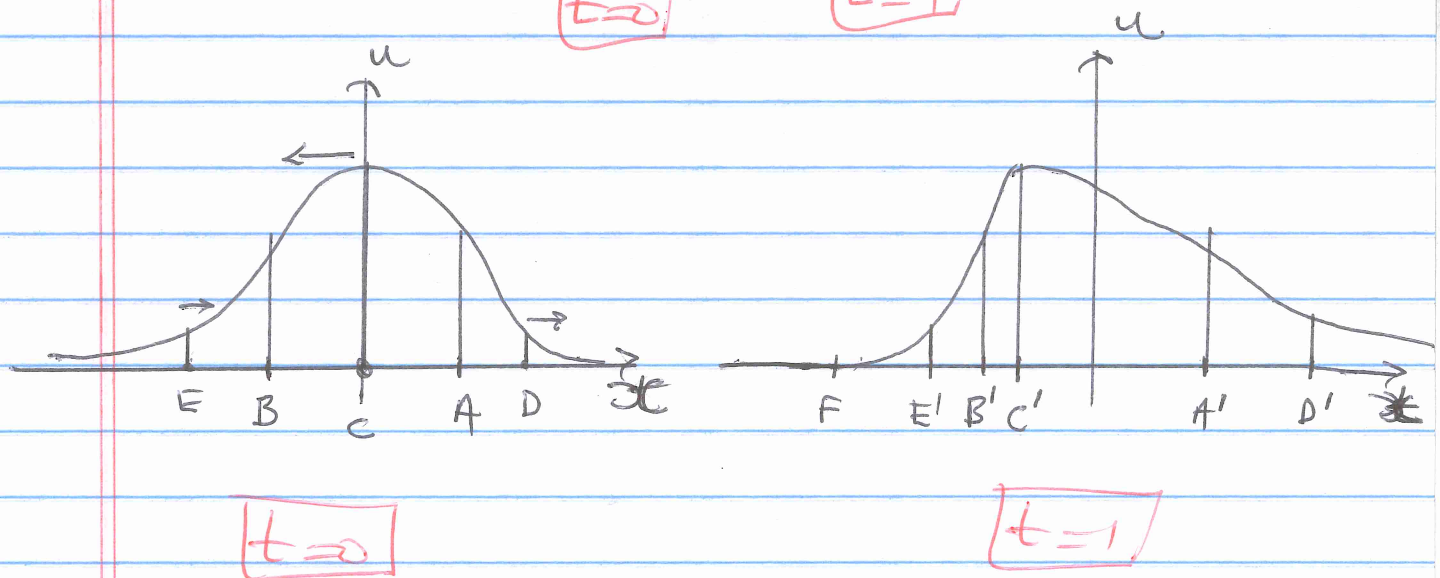


*Below the curve $x = \frac{1+e^{2t}}{1-e^{2t}}$ set $u=0$.
Then VER✓ above this curve we have

$$u(t,x) = \exp\left[-\left(\frac{x+1+(x-1)e^{-2t}}{x+1-(x-1)e^{-2t}}\right)^2\right] \quad (5)$$

–See HWK and previous EX.


x
t
D'
D
+1
A
A'
C
C'
-1
B
B'
E'
E
F
$x = \frac{1+e^{2t}}{1-e^{2t}}$
t=0
t=1


u
x
E
B
C
A
D
t=0
u
x
F
E'
B'
C'
A'
D'
t=1

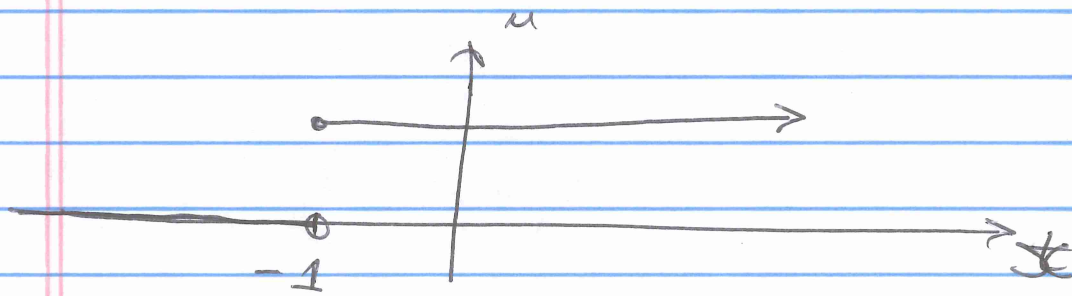
What happens as $t \to \infty$?

- As $t \to \infty$, Region B fills up all of $x < -1$ and $u = 0$ there
- For $x > -1$ by 5

$$\lim_{t \to +\infty} u(t,x) = \exp\left[-\left(\frac{x+1}{x+1}\right)^2\right] = e^{-1}$$

- For $x = -1$, $u(t,-1) = u(0,-1) = e^{-1}$.

So



GENERAL PROPERTIES

1. $\exists!$ CC thru each $(t,x) \in \mathbb{R}^2$
2. CCs cannot cross each other
3. If $t = \beta(x)$ is a CC so are its horizontal translates $t = \beta(x) + k$
4. Each non-horizontal CC is graph of a strictly monotone function.

So each point on a wave moves in same direction $\forall t$

5 As $t \uparrow$ either

a $x(t) \to x_*$ with $c(x_*) = 0$

b $x(t) \to \pm\infty$ in finite time

c $x(t) \to \pm\infty$ in infinite time.

BACK TO ORIGINAL PROBLEM of NONUNIFORM TRANSPORT:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(c(x)\, u\right) = 0$$

ie $\quad u_t + c u_x + c' u = 0. \qquad (6)$

Let $x = x(t)$ by a C.C. and let

$h(t) = u(t, x(t))$ be values of u along it.

Then

$$h'(t) = \frac{\partial u}{\partial t}(t, x(t)) + \frac{\partial u}{\partial x}(t, x(t))\frac{dx}{dt}$$

$$= \frac{\partial u}{\partial t}(t, x(t)) + \frac{\partial u}{\partial x}(t, x(t)) \cdot c(x(t))$$

$$\overset{(6)}{=} -c'(x(t))\, u(t, x(t)) = -c'(x(t))\, h(t)$$

So $$h'(t) = -c'(x(t))\,h(t)$$

SOLN $$h(t) = h(0)\exp\left[-\int_{s=0}^{t} c'(x(s))\,ds\right]$$

OR $$\boxed{u(t,x) = u(0,x_0)\exp\left[-\int_{s=0}^{t} c'(x(s))\,ds\right]} \quad (7)$$

THINK FIX (t,x).

Choose $x = x(s)$ to solve

$$\begin{cases} \dfrac{dx}{ds} = c(x(s)) \\ x(t) = x \end{cases}$$

Let $$\boxed{\beta(x) = \int_0^x \frac{dx}{c(x)}} \quad (8)$$ G.S. $\beta(x(s)) = s + k$

SET $k = \beta(x) - t$

Then $$\boxed{\begin{aligned} x(s) &= \beta^{-1}(s+k) = \beta^{-1}(s + \beta(x) - t) \quad (9) \\ x_0 = x(0) &= \beta^{-1}(\beta(x) - t) \quad (10) \end{aligned}}$$

SOLN given by (7) with (8) – (10).