THE FOURIER TRANSFORM METHOD FOR SOLVING PDEs

[JB, L11]

DEF

① $f \in L^1(\mathbb{R})$ means $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$.

② $f \in L^2(\mathbb{R})$ means $\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty$.

$$\|f\|_{L^2(\mathbb{R})} := \left[\int_{-\infty}^{\infty} |f(x)|^2\,dx\right]^{1/2} \quad \underline{L^2\text{-NORM}}$$

DEF Let $f \in L^1(\mathbb{R})$. The <u>FOURIER TRANSFORM</u> of f is the function

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\,dx \qquad ①$$

WRITE $\hat{f} = F(f)$

NOTES

① $\hat{f} : \mathbb{R} \to \mathbb{C}$.

② F is a <u>LINEAR</u> transformation:

$$F(cf + g) = c\,F(f) + F(g)$$

③ $f \in L^1 \Rightarrow \hat{f}$ IS BOUNDED. Since

$$|\hat{f}(\omega)| \leq \int_{-\infty}^{\infty} |f(x)\, e^{i\omega x}|\,dx = \int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

In fact

ⓐ If $f \in L^1$ Then $\hat{f}$ is uniformly CTS as f^n of ω

ⓑ If $f \in L^1$ Then $|\hat{f}(\omega)| \to 0$ as $\omega \to \pm\infty$.

④ Since θ in $e^{i\theta}$ must be UNITLESS,

units of $\omega = \dfrac{1}{\text{units of } x}$.

EG x = time in sec
ω = frequency in Hz = Cycles/sec

THM You can recover f from $\hat{f}$ using the inverse Fourier transform:

$$f(x) = \left[\mathcal{F}^{-1}(\hat{f}(\omega))\right](x)$$

$$\overset{*}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega x}\, d\omega$$

NOTE
$*$ expresses f as sum (integral) of $\mathbb{C}$ exps with weights $\hat{f}(\omega) = r(\omega)\, e^{i\theta(\omega)}$, with $r(\omega)$ = Amplitude and $\theta(\omega)$ = Phase Shift.

P

DEF If $f, g \in L^1(\mathbb{R})$ Then the <u>CONVOLUTION</u> of f and g is

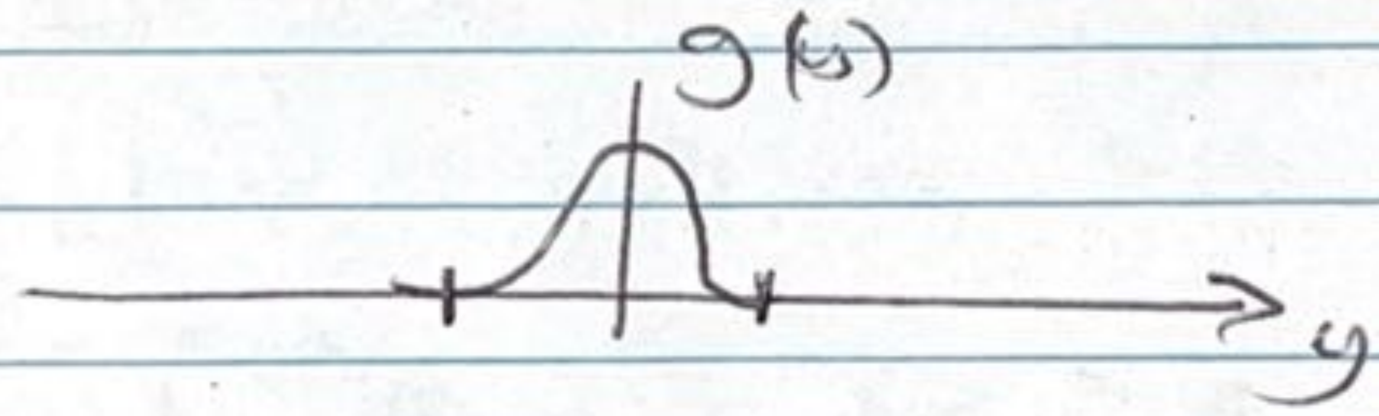
$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

<u>FACTS</u>

① $f, g \in L^1(\mathbb{R}) \Rightarrow f * g \in L^1(\mathbb{R})$

② $f * g = g * f$

③ INTUITION: If g is compactly supported



Then $(f*g)(x) = \int \underbrace{g(y)}_{\uparrow \text{ WEIGHTS}} \; \underbrace{f(x-y)}_{\uparrow \text{ SHIFTS.}} \, dy$

is a weighted sum of values of f near x.

FORMULAE FOR MANIPULATING FOURIER TRANSFORMS

(A) $F[f^{(n)}](\omega) = (-i\omega)^n F[f](\omega)$

F.T. TURNS DIFFERENTIATION into MULT by $-i\omega$

PF $F(f')(\omega) = \int_{\mathbb{R}} f'(x) e^{i\omega x} dx$

$\overset{\text{PARTS}}{=} \left[f(x) e^{i\omega x} \right]_{x=-\infty}^{x=+\infty} - \int f(x)(i\omega) e^{i\omega x} dx$

$u = e^{i\omega x}$
$v' = f'$
$u' = i\omega u$
$v = f$

(B) $F[f(x-a)] = e^{i\omega a} F[f(x)]$

$F^{-1}[e^{i\omega a}\hat{f}(\omega)] = f(x-a)$ SHIFT FORMULA

(C) IF $g(x) = \frac{1}{a} f\left(\frac{x}{a}\right)$ Then $\hat{g}(\omega) = \hat{f}(a\omega)$

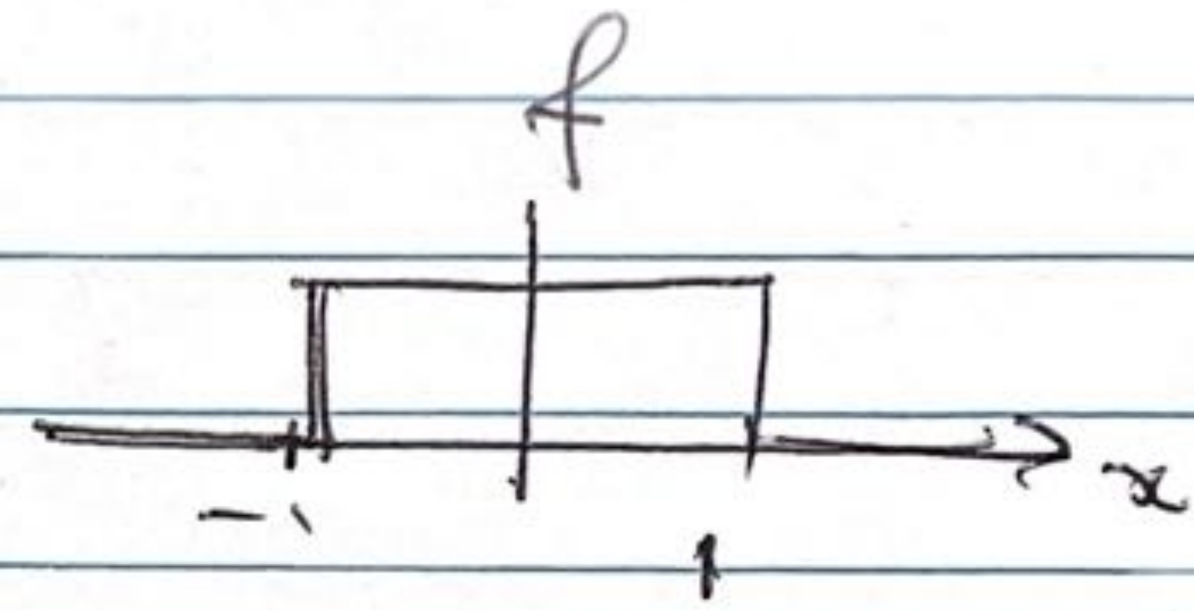
(D) CONVOLUTION THM (~~HARD~~)

$$F^{-1}[\hat{f}(\omega)\,\hat{g}(\omega)] = (f*g)(x)$$

EXS (1) RECTANGLE FN

LET $f(x) = H(1-|x|)$

$f \in L^1(\mathbb{R})$

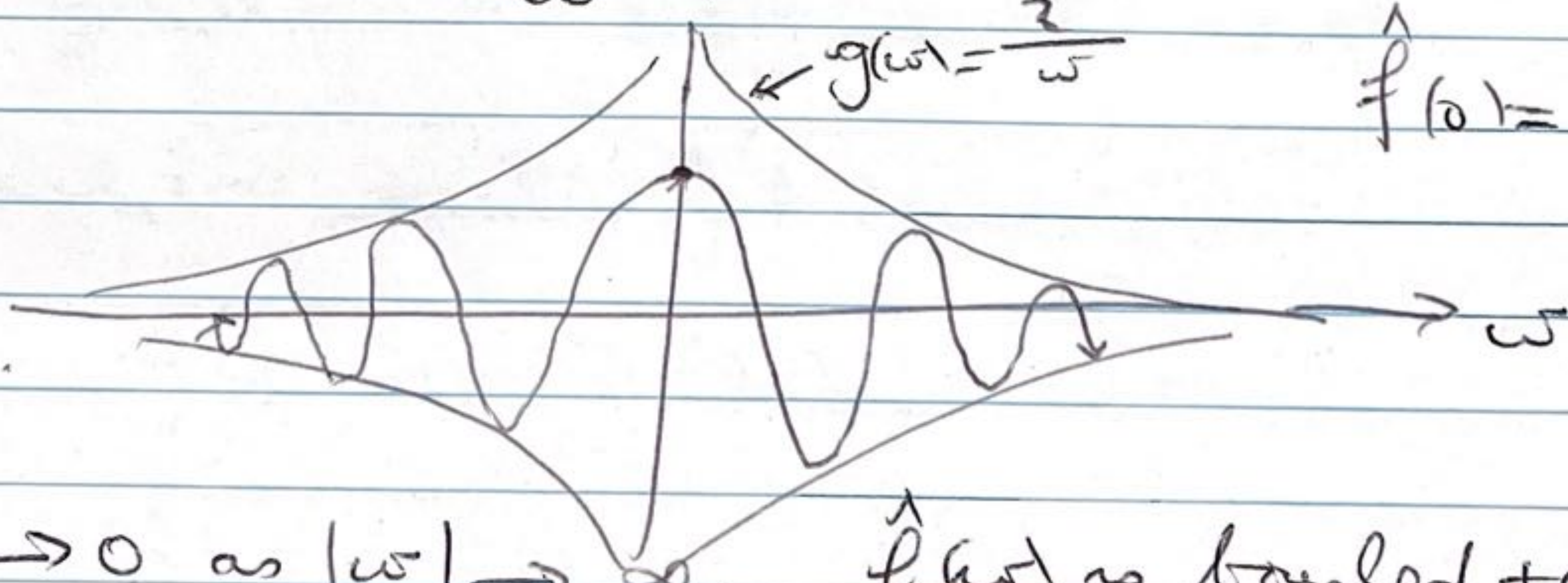


Then

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{i\omega x}\,dx$$

$$= \int_{-1}^{1} e^{i\omega x}\,dx = \frac{e^{i\omega} - e^{-i\omega}}{i\omega}$$

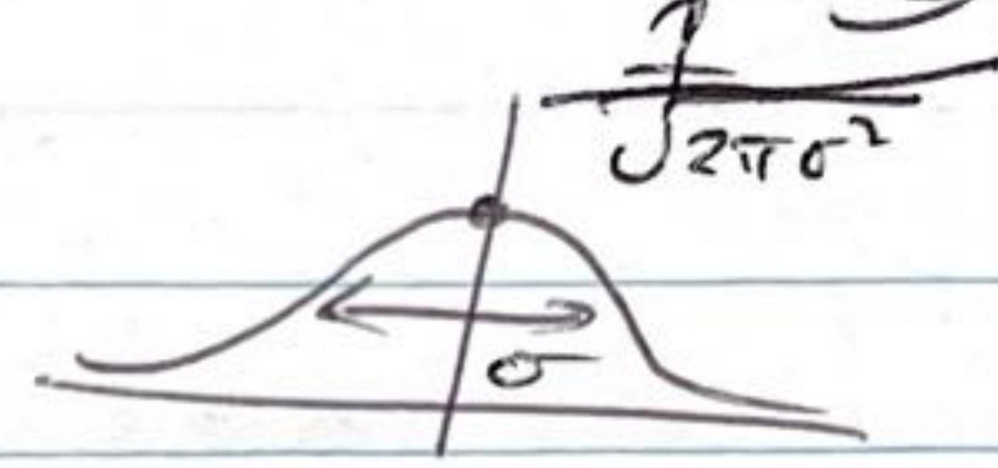
$$= \frac{2\sin(\omega)}{\omega} = 2\,\mathrm{sinc}(\omega)$$



$\hat{f}(0) = 2$

$\hat{f}(\omega) \to 0$ as $|\omega| \to \infty$, $\hat{f}(\omega)$ is bounded + CTS.

2 LET $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$



The $\hat{f}(\omega) = e^{-\sigma^2\omega^2/2}$

PF

CLAIM If $g(x) = e^{-x^2}$ Then $\hat{g}(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$

Given Claim:

$$f(x) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2}\sigma}\, g\left(\frac{x}{\sqrt{2}\sigma}\right)$$

So by ⓒ

$$\hat{f}(\omega) = \frac{1}{\sqrt{\pi}}\, \hat{g}\left(\sqrt{2}\,\sigma\,\omega\right) = e^{-2\sigma^2\omega^2/4} = e^{-\sigma^2\omega^2/2}$$

$\boxed{\text{PF OF CLAIM}}$

Notice $g'(x) = -2x\, e^{-x^2} = -2x\, g(x)$

So $\widehat{g'}(\omega) = -2\, \widehat{x\, g(x)} = 2i\, \widehat{ix\, g(x)}$

$-i\omega\, \hat{g}(\omega) = 2i\, \hat{g}'(\omega)$ [NOTE SIGN ON RHS]

$\hat{g}'(\omega) = -\frac{\omega}{2}\, \hat{g}(\omega)$ ODE

SOLN $\hat{g}(\omega) = \hat{g}(0)\, e^{-\omega^2/4}$

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NOW $\hat{g}(0) = \int_{\mathbb{R}} g(x) e^{i0x} dx = \int g(x)\, dx$

$= \int_{\mathbb{R}} e^{-x^2} dx$

$= \sqrt{\pi}$

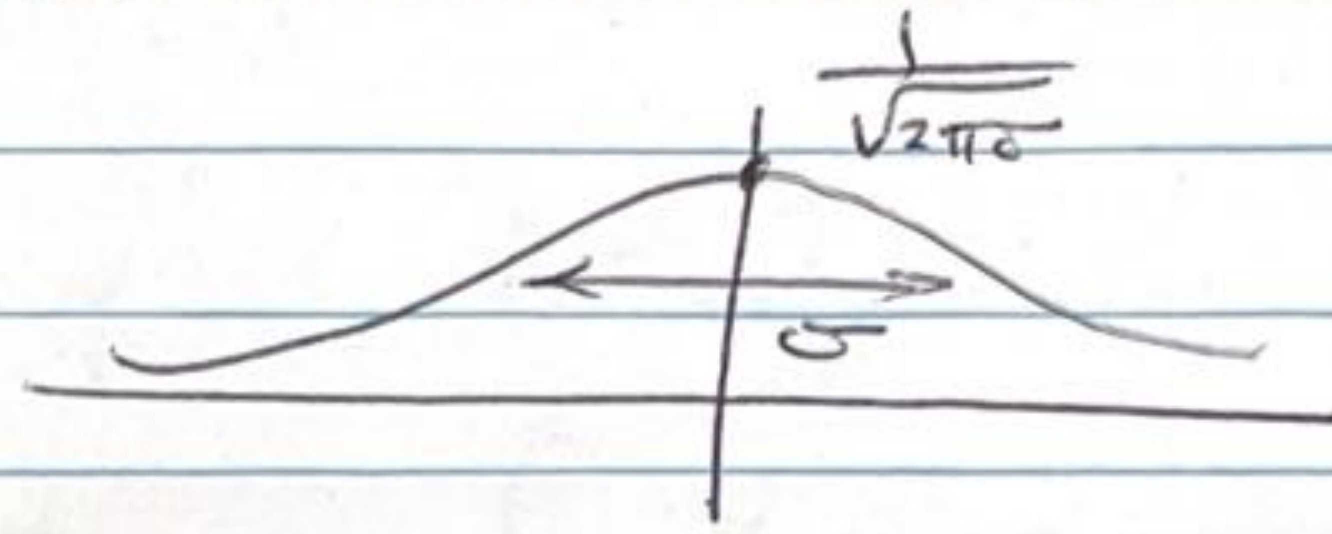
So

$\hat{g}(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$ D.

ALTERNATE PROOF

(2) LET $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

be GAUSSIAN.



Then

$$\hat{f}(\omega) = e^{-\sigma^2\omega^2/2}$$ IS ALSO A GAUSSIAN.

PROOF

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} e^{-x^2/2\sigma^2} e^{i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} e^{-(x - i\sigma^2\omega)^2/2\sigma^2} e^{-\sigma^2\omega^2/2} \, dx$$

by completing the square

$$\overset{*}{=} e^{-\sigma^2\omega^2/2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{-x^2/2\sigma^2} \, dx$$

$$= e^{-\sigma^2\omega^2/2} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \, dy$$

by $y = \frac{x}{\sqrt{2}\sigma}$

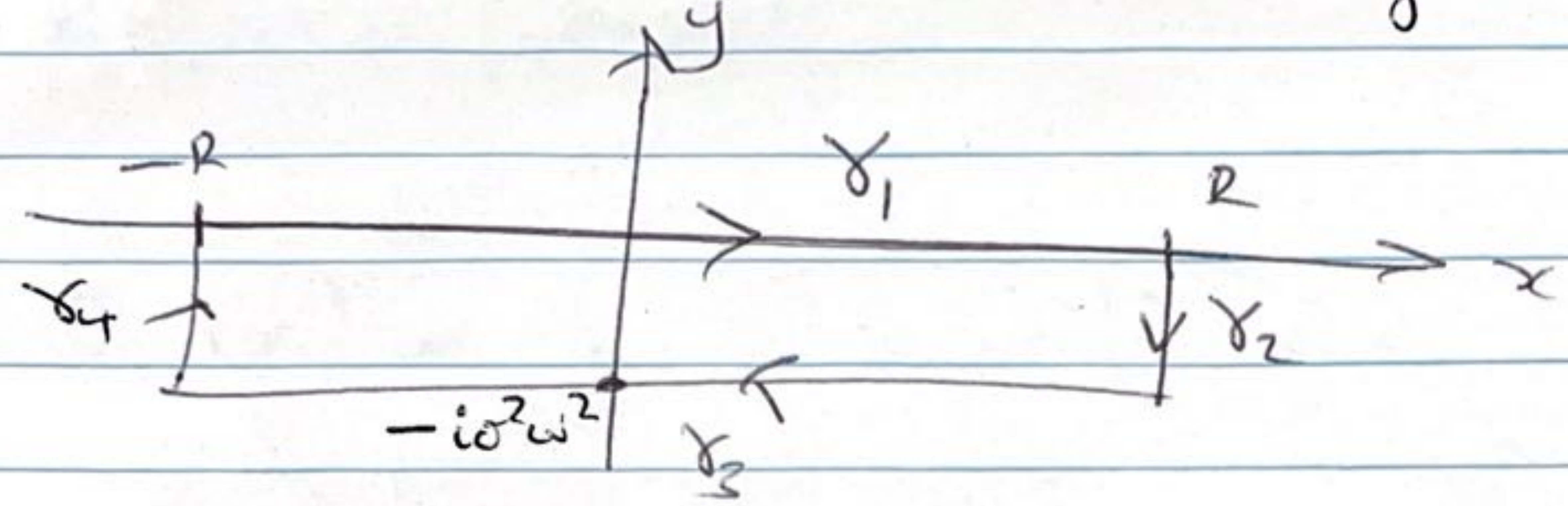
$$= e^{-\sigma^2\omega^2/2} \cdot \underline{1}$$

To prove ⊛ use Cauchy's Theorem from $\mathbb{C}$ Analysis:

(a) Let $f(z) = e^{-z^2/2\sigma^2}$. f is $\mathbb{C}$ Analytic on $\mathbb{C}$

So $\forall$ closed curves γ in $\mathbb{C}$, $\int_\gamma f(z)\,dz = 0$.

(b) PICS



$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

(c) As $R \to \infty$: $\int_{\gamma_2+\gamma_4} f(z)\,dz \longrightarrow 0$

as when $z = \pm R \mp i\sigma^2\omega^2 y \qquad 0 \le y \le 1$

we have

$$|f(z)| = e^{-R^2} \to 0 \quad \text{as } R \to \infty.$$

(d) γ_3: $z(t) = t - i\sigma^2\omega^2$, $-R \le t \le R$

So $dz = dt$

$$\int_{\gamma_3} e^{-z^2/2\sigma^2}\,dz = \int e^{-(t - i\sigma^2\omega^2)^2/2\sigma^2}\,dt$$

We can use F.T. to solve LINEAR PDEs when spatial variable $x \in \mathbb{R}$.

EX 1

$$\begin{cases} u_t = D u_{xx} & x \in \mathbb{R}, \quad t > 0 \\ u(0,x) = f(x) \end{cases}$$

LET

$$\hat{u}(t,\omega) = \int_{\mathbb{R}} u(t,x) e^{i\omega x} dx$$

F.T. WRT SPATIAL VARIABLE x.

Then by Property (A):

$$\hat{u}_t = D(-i\omega)^2 \hat{u}$$

~~SYSTEM OF ODEs~~

$$\boxed{\frac{\partial \hat{u}}{\partial t}(t,\omega) = -\omega^2 D \hat{u}(t,\omega)}$$

– FOR EACH VALUE of parameter ω have ODE in t.

SOLUTION

$$\hat{u}(t,\omega) = e^{-\omega^2 D t} \hat{u}(0,\omega)$$

$$\boxed{\hat{u}(t,\omega) = e^{-\omega^2 D t} \hat{f}(\omega)} \quad \text{by I.C.}$$

So $\hat{u}(t,\omega) = \hat{f}(\omega)\, \hat{g}(t,\omega)$ is a PRODUCT

where $\hat{g}(t,\omega) = e^{-\omega^2 D t}$

So by CONVOLUTION THM

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$$u(t,x) = (f * g)(x) = \int_{\mathbb{R}} f(y)\, g(t, x-y)\, dy$$

TO FIND $g(t,x)$

Recall

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \iff \hat{f}(\omega) = e^{-\sigma^2\omega^2/2}$$

$$\hat{g}(t,\omega) = e^{-\omega^2 Dt}$$

So need $\sigma^2 = 2Dt$

giving $$\boxed{g(t,x) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}}$$

which is FUNDAMENTAL SOL^n TO HEAT EQN

So

$$u(t,x) = \int_{\mathbb{R}} f(y) \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}}\, dy \quad \checkmark$$

EX 2 Wave Eq^n

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \mathbb{R},\ t > 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

Let $\hat{u}(t,\omega) = \int_{\mathbb{R}} u(t,x)\, e^{i\omega x}\, dx$

Get

$$\begin{cases} \hat{u}_{tt}(t,\omega) = -c^2\omega^2 \hat{u}(t,\omega) \\ \hat{u}(0,\omega) = \hat{f}(\omega) \\ \hat{u}_t(0,\omega) = \hat{g}(\omega) \end{cases}$$

ODE IN t WITH PARAMETER ω

So

$$\hat{u}(t,\omega) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t)$$

$$\hat{f}(\omega) = \hat{u}(0,\omega) = A(\omega)$$

$$\hat{g}(\omega) = \hat{u}_t(0,\omega) = c\omega B(\omega)$$

So

$$\hat{u}(t,\omega) = \hat{f}(\omega)\cos(c\omega t) + \frac{\hat{g}(\omega)}{c\omega}\sin(c\omega t)$$

Use $\cos(c\omega t) = \dfrac{e^{ic\omega t} + e^{-ic\omega t}}{2}$

$$\sin(c\omega t) = \frac{e^{ic\omega t} - e^{-ic\omega t}}{2i}$$

to get

$$\hat{u}(t,\omega) = \frac{1}{2}\left[\hat{f}(\omega) + \frac{1}{c\omega i}\hat{g}(\omega)\right] e^{ict\omega}$$
$$+ \frac{1}{2}\left[\hat{f}(\omega) - \frac{1}{c\omega i}\hat{g}(\omega)\right] e^{-ict\omega}$$

$$\hat{u}(t,\omega) = \hat{\phi}_+(\omega) e^{ict\omega} + \hat{\phi}_-(\omega) e^{-ict\omega}$$

SO BY SHIFT FORMULA:

$$u(t,x) = \phi_+(x-ct) + \phi_-(x+ct)$$

CLAIM
$$F^{-1}\left[\frac{\hat{g}(\omega)}{i\omega}\right] = \mp \int_{-\infty}^{x} g(y)\,dy$$

GIVEN CLAIM $\phi_\pm(x) = \frac{1}{2}\left[f(x) \mp \frac{1}{c}\int_{-\infty}^{x} g(y)\,dy\right]$

So recover D'Alembert's Sol^n

$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

PROOF OF CLAIM

Let $h(x) = \int_{-\infty}^{x} g(y)\, dy$

Then $h'(x) = g(x)$

So

$$\hat{g}(\omega) = \hat{h'}(\omega) = -i\omega\, \hat{h}(\omega)$$

as $\hat{h'}(\omega) = \int h'(x)\, e^{i\omega x}\, dx$

$$\overset{\text{PARTS}}{=} -\int h(x)\, i\omega\, e^{i\omega x}\, dx$$

$$= -i\omega\, \hat{h}(\omega)$$

So

$$\hat{h}(\omega) = -\frac{\hat{g}(\omega)}{i\omega}$$

OR

$$F^{-1}\left(\frac{\hat{g}(\omega)}{i\omega}\right)(x) = -\int_{-\infty}^{x} g(y)\, dy$$

$\square$