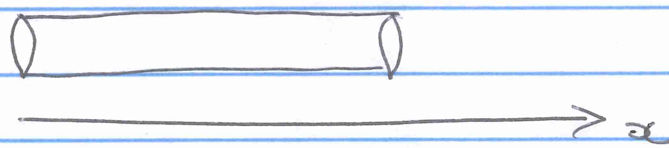
# §2 LINEAR AND NONLINEAR WAVES

## 2.2 THE TRANSPORT EQUATION

Let $u$ be the concentration of a substance in a fluid. (eg a colored dye).

Suppose the fluid is flowing ~~thru~~ a thin pipe with velocity $c$ and that the concentration is constant in each circular cross section.



So $u = u(t, x)$.

CLAIM $u$ satisfies the TRANSPORT EQUATION

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0.$$

NOTE Here $c$ ~~can be~~ is a constant ~~or $c = c(x)$ or $c = c(x,t)$~~.

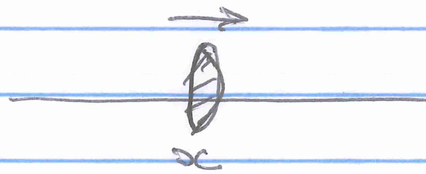
PROOF

IF UNITS of $u$ = MASS/LENGTH

Then

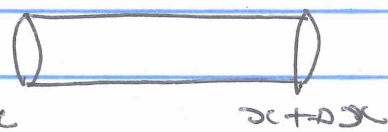
$$\text{MASS in } [x, x+\Delta x] \text{ AT TIME } t = \int_x^{x+\Delta x} u(t,y)\,dy$$

Let $q = q(t, x)$ = Rate at which mass of substance crosses $x$ in +ve direction at time $t$




UNITS of $q$ = MASS/TIME.

CONSIDER a PIPE ELEMENT




Suppose only way substance can enter/leave pipe element is via the ends (no sources or sinks)

By LAW of CONSERVATION of MASS

Rate of Change of Mass in $[x, x+\Delta x]$

= Rate at which mass enters at $x$
− Rate at which mass leaves at $x+\Delta x$

[NOTE: These rates could be negative]

IE
$$\frac{d}{dt}\int_{x}^{x+\Delta x} u(t,y)\,dy = q(t,x) - q(t,x+\Delta x)$$

Take time derivative inside y integral

Apply FTC

$$\int_{x}^{x+\Delta x} \frac{\partial u}{\partial t}(t,y)\,dy = -\int_{x}^{x+\Delta x} \frac{\partial q}{\partial y}(t,y)\,dy$$

OR
$$\int_{x}^{x+\Delta x} \left[\frac{\partial u}{\partial t}(t,y) + \frac{\partial q}{\partial y}(t,y)\right] dy = 0.$$

Since this holds $\forall x, \Delta x$ it is reasonable to conclude

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0}$$

CONSERVATION LAW

NEXT we need a "CONSTITUTIVE LAW" to relate $q$ to $u$:

$$q(t,x) = c(t,x)\,u(t,x)$$

$$\frac{\text{MASS}}{\text{TIME}} = \frac{\text{LENGTH}}{\text{TIME}}\;\frac{\text{MASS}}{\text{LENGTH}}.$$

So
$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(cu) = 0} \qquad (0)$$

If c is constant, then we get
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad \checkmark$$

$\square$.

To obtain a unique solution we pose the IVP:
$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 & \text{for } t > 0 \text{ and } x \in \mathbb{R} \\ u(0, x) = f(x) & \text{for } x \in \mathbb{R} \end{cases} \qquad (1)$$

GIVEN INITIAL CONCENTRATION

$x$ = Position in fixed coordinate frame

$\xi = x - ct$ = Position relative to observer moving with velocity c.

Change of coordinates: $(t, x) \longrightarrow (t, \xi)$

$$v(t, \xi) := u(t, x)$$

OR
$$\boxed{u(t, x) = v(t, x - ct)} \qquad (2)$$

By Chain Rule

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}$$

So

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \left(\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial \xi}\right) + c\frac{\partial v}{\partial \xi}$$

$$= \frac{\partial v}{\partial t}.$$

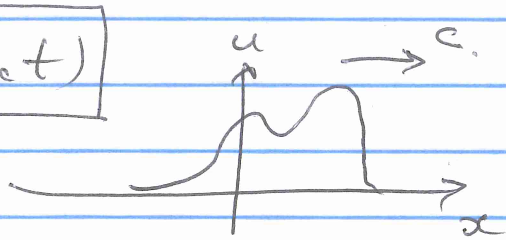
So we need to solve

$$\begin{cases} \dfrac{\partial v}{\partial t} = 0 \\ v(0,\xi) = f(\xi) \end{cases} \qquad (3)$$

SOLUTION $\quad v(t,\xi) = f(\xi)$

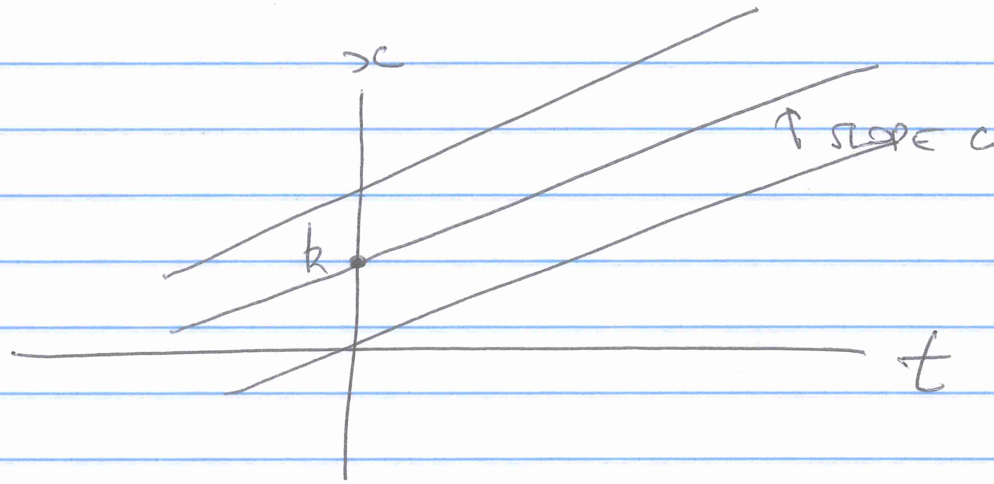
So by (2) $\quad \boxed{u(t,x) = f(x-ct)}$

FIXED $t$




SOLUTION is a TRAVELLING WAVE

- Shape does not change with time
- If $c>0$, Wave moves right with speed $c$
- If $c<0$ ——— left ——— $|c|$.




Since $u$ only depends on

$$\xi = x - ct = \text{"characteristic variable}$$

solution is <u>constant</u> on characteristic lines with slope $c$, ie on lines

$$x = ct + k, \qquad k \in \mathbb{R}$$

- If you know solution on $x$-axis $(t=0)$ then you know solution $\forall (t,x) \in \mathbb{R}^2$.
- The signal propagates along the characteristics

## TRANSPORT WITH DECAY

LET $a > 0$.

consider eqn

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} + au = 0. \\ u(0,x) = f(x) \end{cases}$$

- Models transport of a substance that is decaying exponentially.

Set $u(t,x) = v(t, x-ct)$ as before.

This time

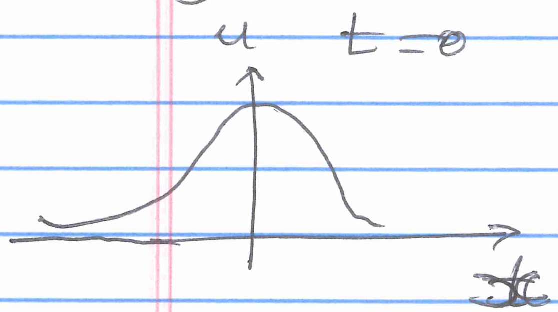
$$\frac{\partial v}{\partial t} + av = 0.$$

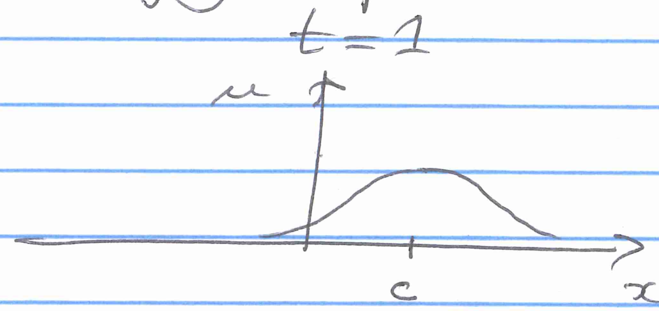
Solution is

$$v(t,\xi) = f(\xi)\, e^{-at}$$

So $\boxed{u(t,x) = f(x-ct)\, e^{-at}}$

Travelling wave with decaying amplitude



## NONUNIFORM TRANSPORT

By 0 or P4 if $c = c(x)$ is not constant

Then

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(cu) = 0$$

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c'u = 0.} \quad (8)$$

**NOTE** Olver studies equations of form

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0.$$

While ~~this~~ may be important in other applications, it does not model transport of a substance in a pipe with nonuniform velocity.

SOLUTION METHOD "THE METHOD OF CHARACTERISTICS"

SIMPLER CASE

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = 0. \quad (9)$$

We can no longer expect $u$ to be constant along lines.

Instead we look for a curve $x = x(t)$ in $(t,x)$-plane along which $u$ is constant

Set

$$h(t) = u(t, x(t))$$

$$h'(t) = u_t(t, x(t)) + u_x(t, x(t))\, x'(t)$$

So if we choose ~~the~~ curve $x = x(t)$ so that

$$\boxed{\frac{dx}{dt} = c(x)} \quad (10)$$

NONLINEAR, SEPARABLE ODE

then

$$h'(t) = u_t + c\, u_x = 0 \qquad \text{by } (9)$$

So $h$ is CONSTANT ALONG THE "CHARACTERISTIC CURVES", $x = x(t)$

$\boxed{\text{CALCULATION OF CHARACTERISTICS}}$

SUPPOSE ~~the~~ function $c$ is never equal to 0. If $c$ is CTS then either $c > 0$ or $c < 0$ everywhere.

By Separation of Variables

$$\int \frac{dx}{c(x)} = \int dt$$

SET

$$\beta(x) = \int_0^x \frac{dy}{c(y)}.$$

INSERT*

$u$ CONST ON CHAR CURVE MEANS $u$ is function of CHAR VAR $\xi = \beta(x) - t$:
$u(t, x) = v(\beta(x) - t)$
for some $f^n$ $v$.

Then

$$\beta(x) = t + k$$

for some constant $k$

Now $\beta' = \frac{1}{c} > 0$ if $c > 0$

So $\beta$ is strictly increasing
So $\beta$ is invertible (and similarly if $c < 0$)

So $$\boxed{x(t) = \beta^{-1}(t + k)} \qquad (11)$$

are characteristic curves.

INSERT*

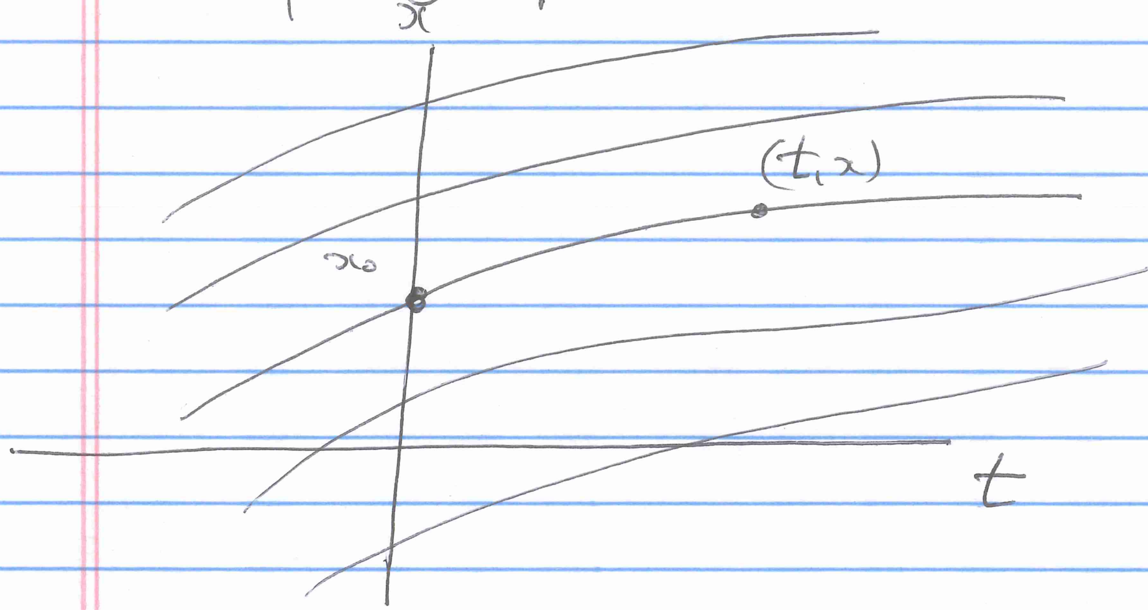
Suppose we want to solve the ~~IVP~~ INITIAL-VALUE PDE problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + c(x) \dfrac{\partial u}{\partial x} = 0 \\ u(0, x) = f(x) \end{cases} \qquad (12)$$

First for each $x_0 \in \mathbb{R}$ we solve The IV ODE problem

$$\begin{cases} \dfrac{dx}{dt} = c(x) \\ x(0) = x_0 \end{cases} \qquad (13)$$

to obtain a family of characteristic curves:




Because of the $\exists!$ THM for solutions of ODEs, these curves do not intersect.

So to find the solution $u$ to (12) at a point $(t, x)$, we find the char. curve passing thru $(t, x)$. If this curve crosses $x$ axis at $x = x_0$, then

$$u(t, x) = u(0, x_0) \quad \text{as } u \text{ CONST ON CHAR CURVE}$$
$$= f(x_0).$$

ON OTHER HAND, if this char curve does not cross $x$ axis, then initial data, $f$, cannot be used to ~~pres~~ give solution at $(t, x)$.

**EX 1** [EX 2.4, p26]

$$c(x) = \frac{1}{x^2+1} > 0$$

**IVP**

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{x^2+1}\dfrac{\partial u}{\partial x} = 0 \\ u(0,x) = \dfrac{1}{1+(x+3)^2} \end{cases} \qquad (14)$$

**CHAR CURVES**

$$\frac{dx}{dt} = \frac{1}{x^2+1}$$

$$\int (x^2+1)\,dx = \int dt$$

$$\boxed{\frac{1}{3}x^3 + x = t + k}$$

$\beta(x) = \frac{1}{3}x^3 + x$

As $k$ varies, get family of curves in $(t,x)$-plane
See Fig 2.7.

**CHAR VARIABLE** $\quad \xi = \beta(x) - t = \frac{1}{3}x^3 + x - t.$

**SOLUTION** of form

$$u(t,x) = v\left(\frac{1}{3}x^3 + x - t\right)$$

for some function $v$

To find $v$ we use initial condition:

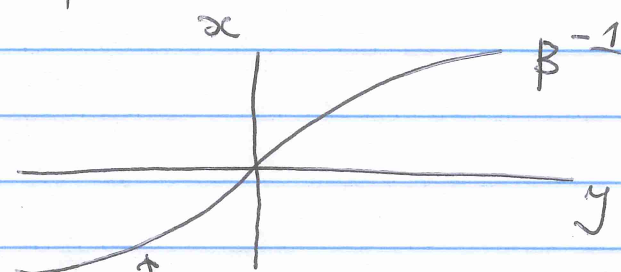
$$u(0, x) = \frac{1}{1+(x+3)^2}$$

$$\Rightarrow \quad v\left(\frac{1}{3}x^3 + x\right) = \frac{1}{1+(x+3)^2}$$

SET $\quad y = \beta(x) = \frac{1}{3}x^3 + x.$

Since $\beta'(x) = x^2 + 1 > 0$, $\beta$ is INVERTIBLE

$$x = \beta^{-1}(y)$$




THIS IS SAME CURVE AS CHAR CURVE THAT GOES THRU (0,0) IN FIG 2.7

and so

$$\boxed{v(y) = \frac{1}{1 + \left[\beta^{-1}(y) + 3\right]^2}}$$

So

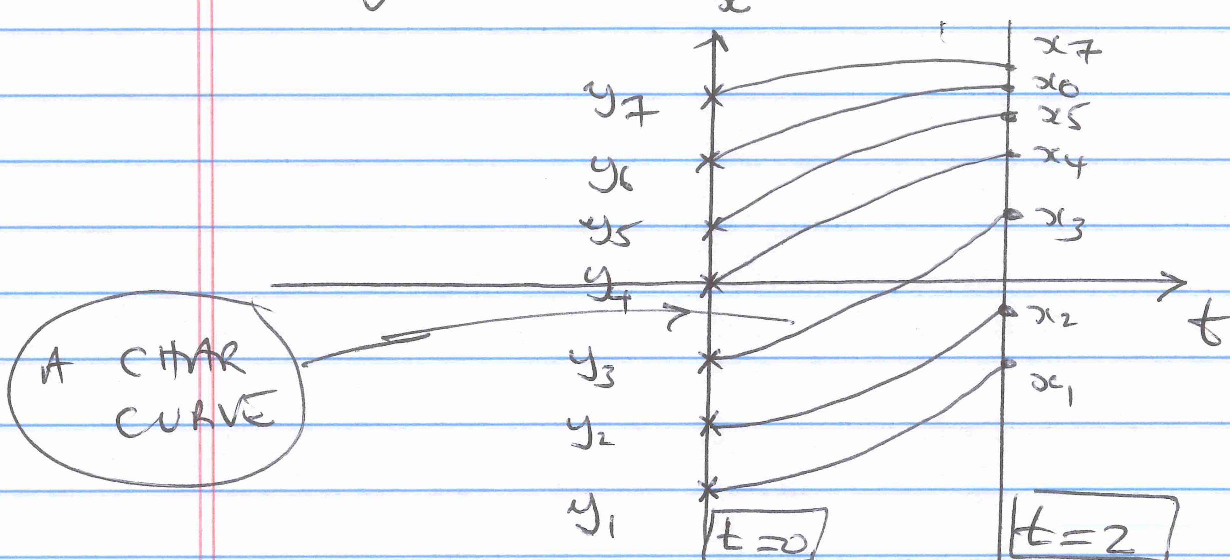
$$u(t, x) = v(\beta(x) - t) = \frac{1}{1 + \left[\beta^{-1}(\beta(x) - t) + 3\right]^2} \qquad (15)$$

where $\beta(x) = \frac{1}{3}x^3 + x$ is SOLUTION to (14).

Since $\exists$ formula for roots of a cubic, in this example $\exists$ formula for $\beta^{-1}$, hence for $u$.

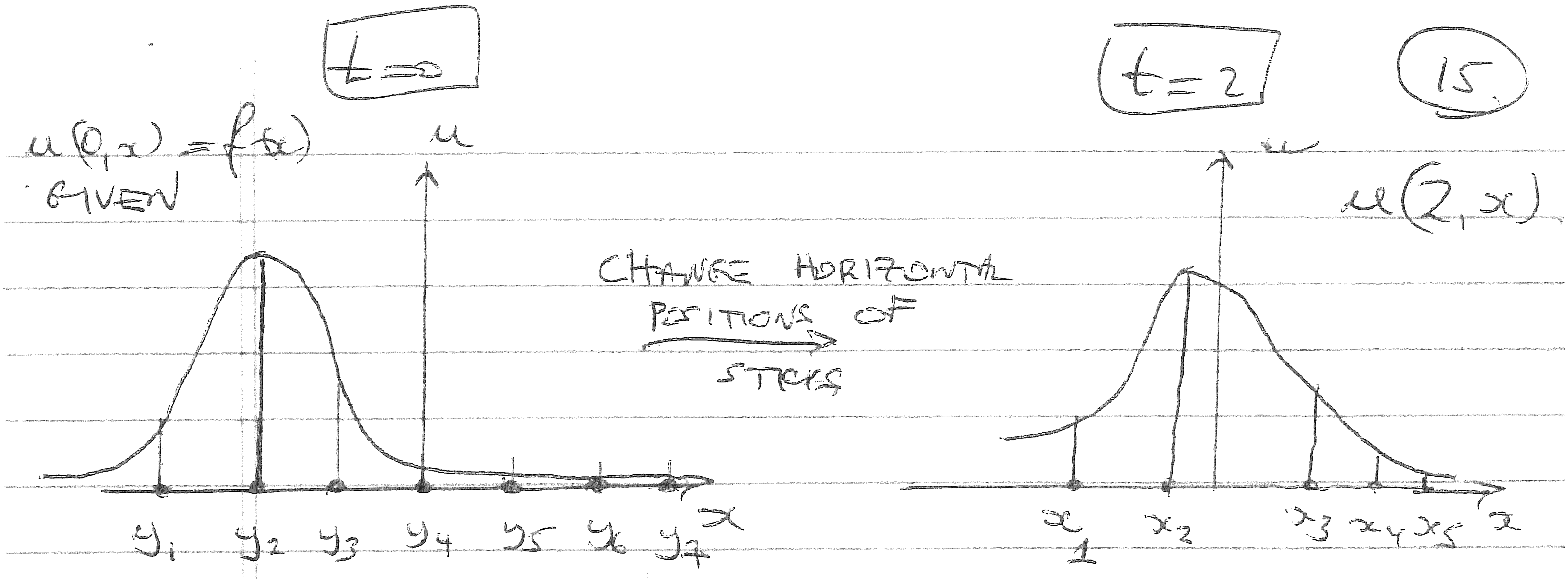
But it is not illuminating!

INSTEAD by hand, or better with MATLAB we can generate snapshots of solution




GIVEN equally spaced points $y_1 - y_N$ on $x$ axis, use ODE solver to calculate char curves thru those points. Stop ODE solver at $t=2$ to get points $x_1, \ldots x_N$.

So $$u(2, x_j) = u(0, y_j) \qquad j = 1, \ldots, N.$$



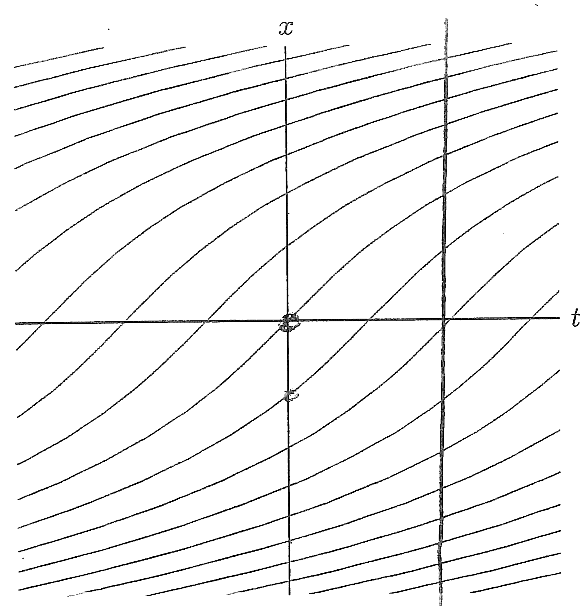




Figure 2.7. Characteristic curves for $u_t + (x^2+1)^{-1} u_x = 0$

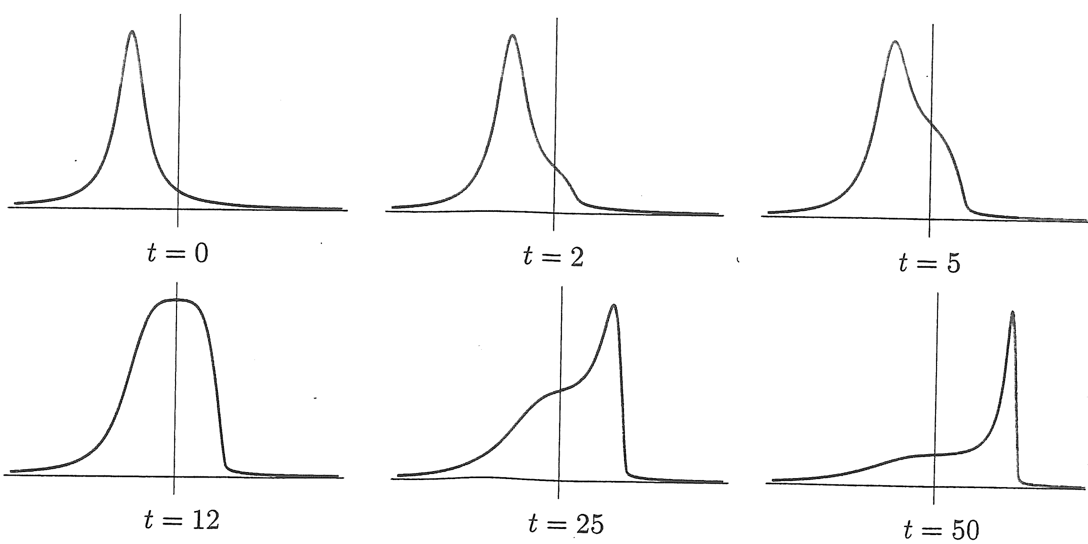



**Figure 2.8.** Solution to $u_t + \dfrac{1}{x^2+1}\, u_x = 0.$