DUHAMEL'S PRINCIPLE FOR HEAT EQN ON $\mathbb{R}$

GOAL: Find solution $u = u(t,x)$ to IVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + F(t,x) & \text{for } x \in \mathbb{R},\ t > 0 \\ u(0,x) = 0 \end{cases} \qquad (A)$$

where $F = F(t,x)$ is a given forcing function.

To solve (A):

For each choice of parameter $\tau > 0$ find $w = w(t,x;\tau)$ satisfying IVP

$$\begin{cases} \dfrac{\partial w}{\partial t} = D\dfrac{\partial^2 w}{\partial x^2} \\ w(0,x;\tau) = F(\tau,x) \end{cases} \qquad (B)$$

NOTE Since I.C. ~~de~~ is a function of x that depends on parameter, τ, the solution $w(t,x;\tau)$ is a function of (t,x) that also depends on the parameter τ.

THM [DUHAMEL'S PRINCIPLE]

Suppose $w = w(t, x; \tau)$ solves (B) for each parameter $\tau > 0$.

Then the solution to (A) is given by

$$\boxed{u(t, x) = \int_0^t w(t - \tau, x; \tau)\, d\tau} \quad (C)$$

NOTE $w(t-\tau, x; \tau)$ shifts the solution $w(t, x; \tau)$ forward in time by τ.

This means that the effect of the force $F(\tau, x)$ that acts as an IC @ time $t = 0$ in (B) is moved forward in time to $t = \tau$ in the solution (C).

DIFFERENTIATION UNDER INTEGRAL THM

Given Functions $x = a(t)$, $x = b(t)$

and $G = G(t,x)$ that are differentiable

Let

$$g(t) = \int_{x=a(t)}^{x=b(t)} G(t,x)\, dx.$$

Then

$$g'(t) = G(t, b(t))\, b'(t) - G(t, a(t))\, a'(t) + \int_{x=a(t)}^{x=b(t)} \frac{\partial G}{\partial t}(t,x)\, dx$$

PROOF IDEAS

① IF $G = G(x)$ is t-independent then FTC + Chain Rule give

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} G(x)\, dx\right] = G(b(t))\, b'(t) - G(a(t))\, a'(t)$$

② IF a, b are constants

$$\frac{\partial}{\partial t} \int_a^b G(t,x)\, dx = \lim_{h \to 0} \frac{\int_a^b G(t+h,x)\, dx - \int_a^b G(t,x)\, dx}{h}$$

$$= \lim_{h\to 0} \int_a^b \frac{G(t+h,x) - G(t,x)}{h}\, dx$$

$$\overset{\circledast}{=} \int_a^b \lim_{h\to 0} \frac{G(t+h,x) - G(t,x)}{h}\, dx$$

$$= \int_a^b \frac{\partial G}{\partial t}(t,x)\, dx.$$

$\circledast$ This step can be justified by the Bounded Convergence Theorem. One version of this says:

If $[a,b]$ is a bounded interval and $f_n : [a,b] \to \mathbb{R}$ is a sequence of integrable functions so that $\exists M : \forall n, \forall x\, |f_n(x)| \le M$.

Then
$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx$$

(In particular $f(x) := \lim_{n\to\infty} f_n(x)$ $\exists$)

PROOF OF DUHAMEL SHOW (C) SOLVES (A)

(1) $u(0,x) = \int_0^0 w(0-\tau, x; \tau)\, d\tau = 0$. by (C)

(2) By (C) and Diff^n Under Integral Thm:

$$\frac{\partial u}{\partial t} = w(t-t; x; t).1 - w(t-0, x; 0)\,0$$

$$+ \int_0^t \frac{\partial}{\partial t}\left(w(t-\tau, x; \tau)\right) d\tau$$

$$\stackrel{(B)}{=} w(0, x, t) + \int_0^t \frac{\partial^2}{\partial x^2} w(t-\tau, x; \tau)\, d\tau$$

as w solves heat eqn

$$= F(t, x) + \frac{\partial^2}{\partial x^2} \int_0^t w(t-\tau, x; \tau)\, d\tau$$

by IC in (B) and Diff^n Under Integral Theorem again. (Notice: Here we differentiate wrt x but integrate wrt τ.)

$$\stackrel{(C)}{=} F(t, x) + \frac{\partial^2 u}{\partial x^2}$$

$\square$

NEXT We say before that solution to

$$(B')\begin{cases} \dfrac{\partial w}{\partial t} = D\dfrac{\partial^2 w}{\partial x^2} \\ w(0,x) = F(x) \end{cases}$$

is $w(t,x) = \int_{\mathbb{R}} S(t, x-y)\, F(y)\, dy$

where

$$S(t,x) = \frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}$$ is FUNDAMENTAL SOL^n.

Only difference between (B), (B') is that IC depends on parameter τ: $F = F(x;\tau)$.

So sol^n to (B) is

$$w(t,x;\tau) = \int_{\mathbb{R}} S(t, x-y)\, F(\tau, y)\, dy \quad (D)$$

Finally using (C)

CONVOLUTION IN t and x

$$\boxed{\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}} S(t-\tau, x-y)\, F(\tau,y)\, dy\, d\tau \\ &= \int_0^t \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi D(t-\tau)}}\, e^{-(x-y)^2/4D(t-\tau)}\, F(\tau,y)\, dy\, d\tau \end{aligned}} \quad (E)$$