[JB, #9] HEAT EQUATION ON $\mathbb{R}$ WITH SOURCES: DUHAMEL'S PRINCIPLE

FORM: SOLVE IVP for $u = u(t,x)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + F(t,x) & x \in \mathbb{R},\ t>0 \\ u(0,x) = 0 \end{cases} \quad (1)$$

Here $F = F(t,x)$ is a given function modeling heat sources/sinks, a "forcing" function

MOTIVATION: from simple forced ODE IVP for $y = y(t)$:

$$\begin{cases} \dfrac{dy}{dt} + ay = F(t) \\ y(0) = 0 \end{cases} \quad (2)$$

DUHAMEL'S PRINCIPLE

For each $\tau > 0$ consider the IVP for $w = w(t;\tau)$

$$\begin{cases} \dfrac{dw}{dt} + aw = 0 \\ w(0) = w(0;\tau) = F(\tau) \end{cases} \quad (3)$$

Then

$$y(t) = \int_0^t w(t-\tau;\tau)\, d\tau \qquad (4)$$

INTUITION

Imagine the source $F(t)$ only acts for an instant (F is IMPULSIVE):

$$F(t) = F(\tau)\,\delta(t-\tau) \qquad \text{for some } \tau.$$

So its like we are solving

$$\begin{cases} \dfrac{dy}{dt} + ay = 0 & t > \tau \\ y(\tau) = F(\tau) \end{cases} \qquad (5)$$

and setting $y(t) \equiv 0$ for $t < \tau$.

OR with $w(t) = y(t+\tau)$:

$$\begin{cases} \dfrac{dw}{dt} + aw = 0 & t > 0 \\ w(0) = F(\tau) \end{cases} \qquad (5')$$

So

$$y(t) = 0 \qquad \text{for } t < \tau$$
$$y(t) = w(t-\tau). \qquad \text{for } t > \tau$$

Instead of an impulsive force, for general source, $F = F(t)$ we have

$$F(t) = \int_{\mathbb{R}} F(\tau)\, \delta(t-\tau)\, d\tau$$

So by Principle of Superposition we would except

$$y(t) = \int_{\tau=0}^{\tau=t} w(t-\tau;\tau)\, d\tau$$

$\boxed{\text{RIGOROUS PROOF I}}$

(a) Using integrating factors solution of (2) is

$$y(t) = \int_0^t e^{-a(t-s)} F(s)\, ds \qquad (6)$$

$$\frac{d}{dt}\left(e^{at} y(t)\right) = e^{at}(ay + y') = e^{at} F(t)$$

So taking $\int_0^t$:

$$e^{at} y(t) = \int_0^t e^{as} F(s)\, ds$$

(b) Sl^n of (3) is

$$w(t) = w(0)\, e^{-at}$$

or $$w(t;\tau) = F(\tau)\, e^{-at}$$

(c) So by (6)

$$y(t) = \int_0^t e^{-a(t-\tau)} F(\tau)\, d\tau$$
$$= \int_0^t w(t-\tau;\tau)\, d\tau \qquad \square$$

$\boxed{\text{RIGOROUS PROOF II}}$

Consider harder problem:

Suppose $\mathcal{H} = \mathbb{R}^n$ or $\mathcal{H} = C^2(\mathbb{R})$ = Twice differentiable f^{n}s on $\mathbb{R}$

We have for each t, $u(t) \in \mathcal{H}$.

So for $\mathcal{H} = \mathbb{R}^n$ $u(t) = \vec{u}(t)$ is a vector

And for $\mathcal{H} = C^2(\mathbb{R})$ $u(t) = u(t,x)$ is f^n of x for each t.

Consider

$$\begin{cases} \dfrac{du}{dt} + \mathcal{L}(u) = F(t) & \text{for } t > 0 \\ u(0) = 0 \in \mathcal{H} \end{cases} \qquad (7)$$

where $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is a linear operator

Case $\mathcal{H} = \mathbb{R}^n$: $\mathcal{L}\vec{u} = A\vec{u}$ for matrix A

Case $\mathcal{H} = C^2(\mathbb{R})$: $\mathcal{L}u = \dfrac{\partial^2 u}{\partial x^2}$

Then solution is obtained by solving for each $\tau > 0$

$$\begin{cases} \dfrac{dw}{dt} + \mathcal{L}(w) = 0 & \text{for } t > 0 \\ w(0; \tau) = F(\tau) \end{cases} \qquad (8)$$

and setting

$$y(t) = \int_0^t w(t - \tau; \tau)\, d\tau \qquad (9)$$

Pf

Use

DIFFERENTIATION UNDER INTEGRAL THM

IF $g(t) = \displaystyle\int_{x=a(t)}^{x=b(t)} f(t, x)\, dx$ Then

$$g'(t) = f(t, b(t)) - f(t, a(t)) + \int_{a(t)}^{b(t)} \frac{\partial G}{\partial t}(t, x)\, dx$$

Show 9 solves 7.

Well $\quad y(0) = 0$ ✓

And by Thm above

$$\frac{dy}{dt} = w(t-t;t) - w(t-0;0) + \int_0^t \frac{\partial w}{\partial t}(t-\tau;\tau)\,d\tau$$

$$= w(0;t) - w(t;0) + \int_0^t -(\mathcal{L}w)(t-\tau;\tau)\,d\tau$$

$$= F(t) - 0^{(*)} - \mathcal{L}\int_0^t w(t-\tau;\tau)\,d\tau$$

by linearity of $\mathcal{L}$

$$= F(t) - \mathcal{L}y \quad ✓$$

(*) As $F(0) = 0$ in 8 gives $w \equiv 0$
$(\tau = 0)$

□

BACK TO 1 (A special case of 7)

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + F(t,x) & \text{①} \\[2ex] u(0,x) = 0 \end{cases}$$

Recall solⁿ of

$$\begin{cases} \dfrac{\partial w}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2} \\[2ex] w(0, x) = F(x) \end{cases}$$

is

$$w(t, x) = \int_{\mathbb{R}} S(t, x-y)\, F(y)\, dy$$

with

$$S(t, x) = \frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}$$

FUNDAMENTAL SOLUTION

So by (8), (9) get solⁿ of (1) is

$$u(t, x) = \int_0^t w(t-\tau, x; \tau)\, d\tau$$

$$\boxed{u(t, x) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D(t-\tau)}}\, e^{-(x-y)^2/4D(t-\tau)}\, F(\tau; y)\, d\tau \qquad (10)}$$

By Principle of ~~Superposition~~ get

THM

Sol^n of IVP for $u = u(t, x)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + F(t, x) & x \in \mathbb{R},\ t > 0 \\ u(0, x) = f(x) & x \in \mathbb{R} \end{cases} \qquad (11)$$

is

$$u(t, x) = \frac{1}{2\sqrt{\pi D t}} \int_{\mathbb{R}} e^{-(x-y)^2/4Dt} f(y)\, dy$$

$$+ \int_0^t \frac{1}{2\sqrt{\pi D(t-\tau)}} \int_{\mathbb{R}} e^{-(x-y)^2/4D(t-\tau)} F(\tau, y)\, dy\, d\tau$$

IF $f \in C(\mathbb{R})$ is bounded and F is twice differentiable (12) in x and once differentiable in t, Then so is u

EXAMPLE where can do integral.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + tx & x \in \mathbb{R},\ t > 0 \\ u(0, x) = 0 & x \in \mathbb{R} \end{cases}$$

By (12)

$$u(t, x) = \int_0^t \tau \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi(t-\tau)}}\, e^{-(x-y)^2/4D(t-\tau)}\, y\, dy\, d\tau$$

Let

$$r = \frac{x-y}{2\sqrt{t-\tau}} \qquad dr = \frac{-dy}{2\sqrt{t-\tau}}$$

$$y = x - 2r\sqrt{t-\tau}$$

So get

$$u(t,x) = \int_0^t \frac{\tau}{\sqrt{\pi}}\left[x\int_{\mathbb{R}} e^{-r^2}\,dr - 2\sqrt{t-\tau}\,\cancelto{0}{\int_{\mathbb{R}} r\,e^{-r^2}\,dr}\right]$$

$$u(t,x) = x\int_0^t \tau\,d\tau = \frac{xt^2}{2}.$$