THEORETICAL RESULTS ABOUT HEAT EQUATION

based on SHEARER + LEVY

CAUCHY PROBLEM ON $\mathbb{R}$

$$\begin{cases} u_t = k\,u_{xx} & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x) \end{cases} \qquad (1)$$

INITIAL BOUNDARY VALUE PROBLEM ON $[a,b]$

$u_t = k u_{xx}$ $\quad a \le x \le b,\ t > 0$ $\quad$ PDE

$u(0,x) = u_0(x)$ $\quad$ INITIAL CONDITION $\qquad (2)$

$$\begin{cases} u(t,a) = g(t) \\ u(t,b) = h(t) \end{cases}$$ DIRICHLET BOUNDARY VALUES

OR

$$\begin{cases} u_x(t,a) = g(t) \\ u_x(t,b) = h(t) \end{cases}$$ NEUMANN BOUNDARY VALUES.

DEF HEAT ENERGY $\quad H(t) = \int_a^b u(t,x)\,dx$

CLAIM I ON A BOUNDED DOMAIN WITH ZERO NEUMANN CONDITIONS, $u_x(t,a) = 0$ and $u_x(t,b) = 0$ THEN HEAT ENERGY IS CONSTANT

NOTE HEAT FLUX $= -ku_x$.
So ZERO NEUMANN CONDITIONS states that no heat flows through ends of interval, ie ends are insulated

PF

$$H'(t) = \int_a^b u_t(t,x)\,dx$$
$$= \int_a^b k u_{xx}(t,x)\,dx$$
$$= \Big[k u_x(t,\cdot)\Big]_{x=a}^{x=b}$$
$$= k[u_x(t,b) - u_x(t,a)] = 0. \quad \square$$

CLAIM II On unbounded domain $\mathbb{R}$
If u is integrable ~~the~~ and $u_x \to 0$ as $x \to \pm\infty$
Then Heat energy is finite.

DEF MATHEMATICAL ENERGY $E(t) = \int_a^b u^2(t,x)\,dx$

CLAIM III ON A BOUNDED DOMAIN WITH EITHER ZERO DIRICHET OR ZERO NEUMANN B.Cs

$$E'(t) \le 0 \quad \forall t \ge 0$$
So $\quad E(t) \le E(0) \quad \forall t > 0$

CLAIM IV The same result is true on unbounded domain $\mathbb{R}$ if $E(t)$ is defined and $u(t,x) \to 0$ as $x \to \pm\infty$ and u_x is bounded in x.

PF OF CLAIM III

$u_t = k u_{xx}$ $\times u$ gives

$$\int_a^b u u_t \, dx = k \int_a^b u u_{xx} \, dx$$

$$\overset{\text{PARTS}}{=} k u u_x \Big|_a^b - k \int_a^b u_x^2 \, dx$$

So

$$\frac{1}{2} E'(t) = \frac{d}{dt} \int_a^b \frac{1}{2} u^2 \, dx$$

$$= \int_a^b u u_t \, dx$$

$$= -k \int_a^b u_x^2 \, dx \leq 0$$

if have either zero Dirichlet / Neumann BCs.

$\square$

UNIQUENESS THM On $[a,b]$ or $\mathbb{R}$:

If u_1, u_2 both solve (2) then $u_1 = u_2$

PF

Let $u = u_1 - u_2$.
Then u solves (2) with ZERO BC's. and ZERO IC.

So by CLAIM III or IV

$$0 \le \int_a^b [u(t,x)]^2 dx \le \int_a^b [u(0,x)]^2 dx$$
$$= \int_a^b 0 \, dx = 0$$

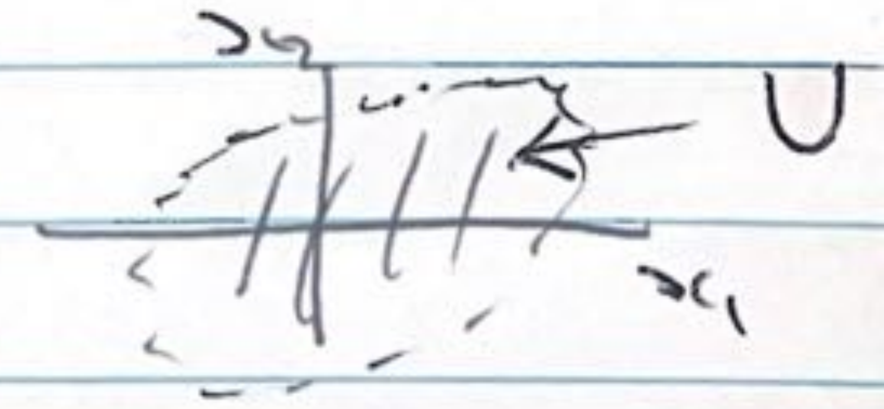
which forces $\int_a^b u^2 dx = 0$ so $u \equiv 0$.

$\square$

MAXIMUM PRINCIPLE

Most naturally formulated in case $\vec{x} = (x_1, \ldots x_n) \in \mathbb{R}^n$

Let U be a bounded open set in $\mathbb{R}^n$



Consider Heat Eq^n

$$u_t = k \Delta u \qquad \vec{x} \in U, \; t > 0$$

where <u>LAPLACIAN</u> of u is

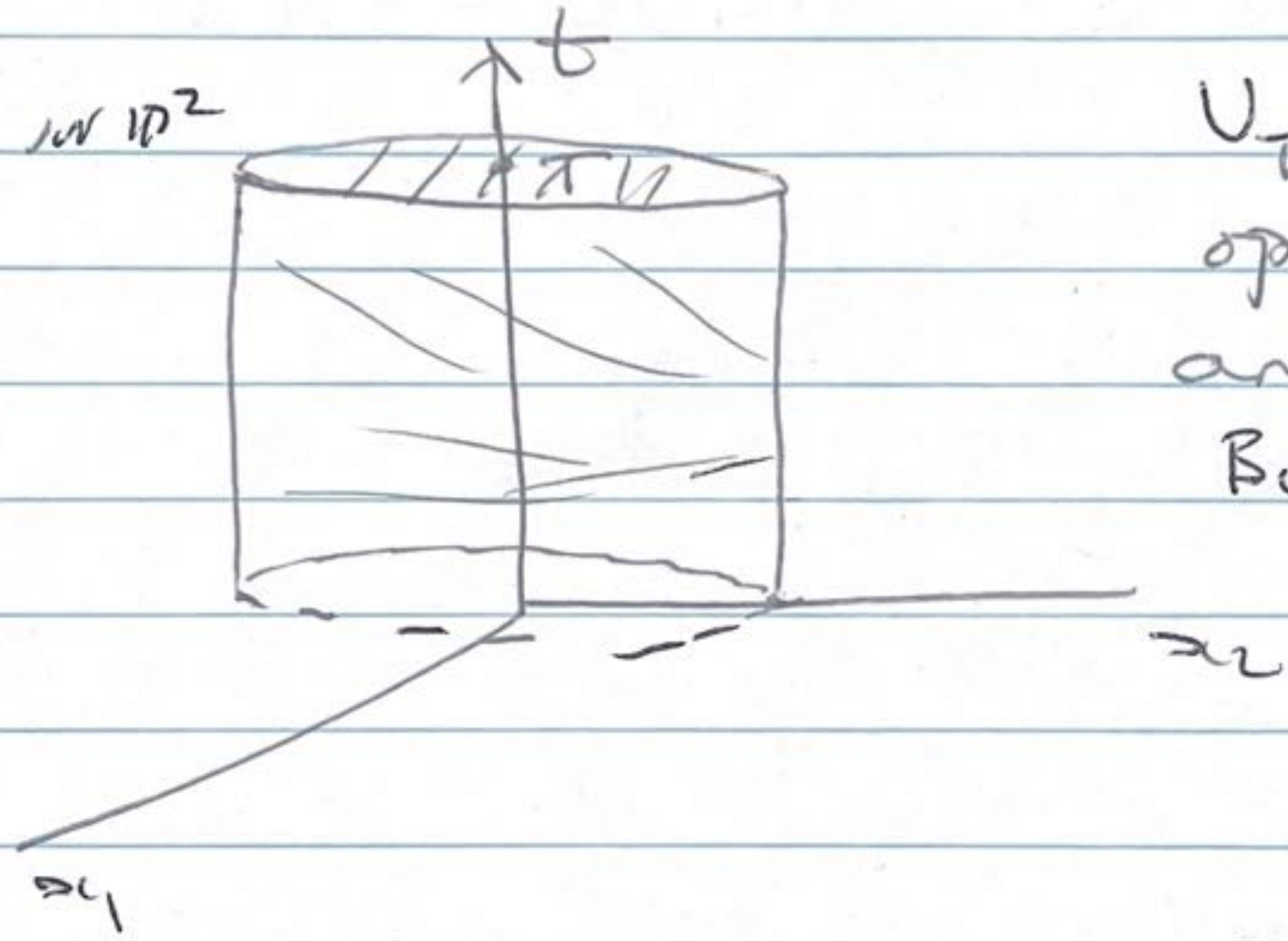
$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}.$$

Fix time $T > 0$
Let

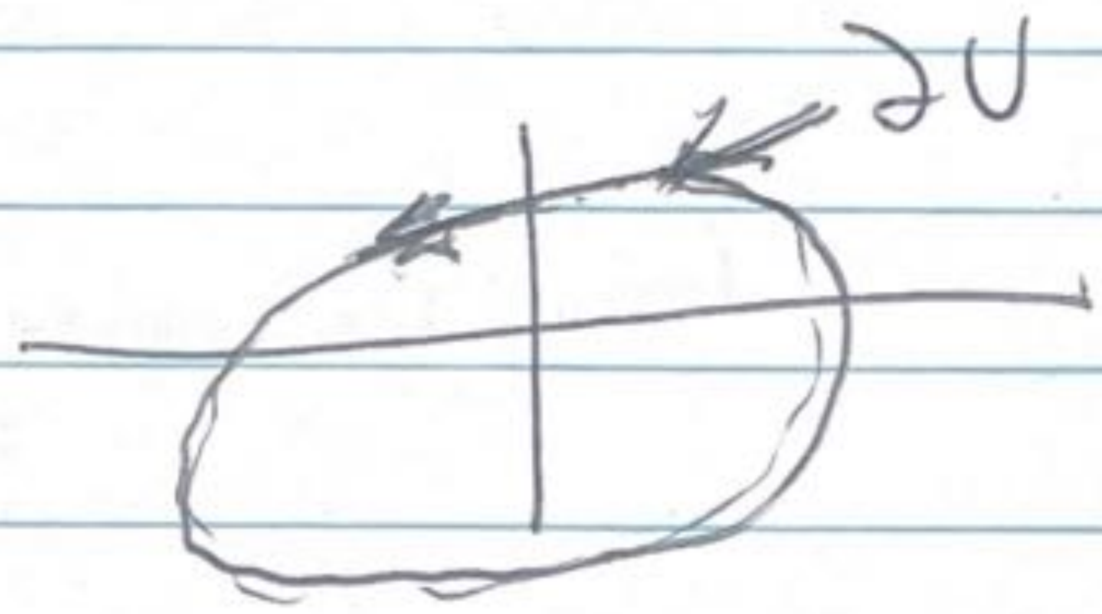
$$U_T = U \times (0, T]$$

PICTURE

IF U = DISC IN $\mathbb{R}^2$



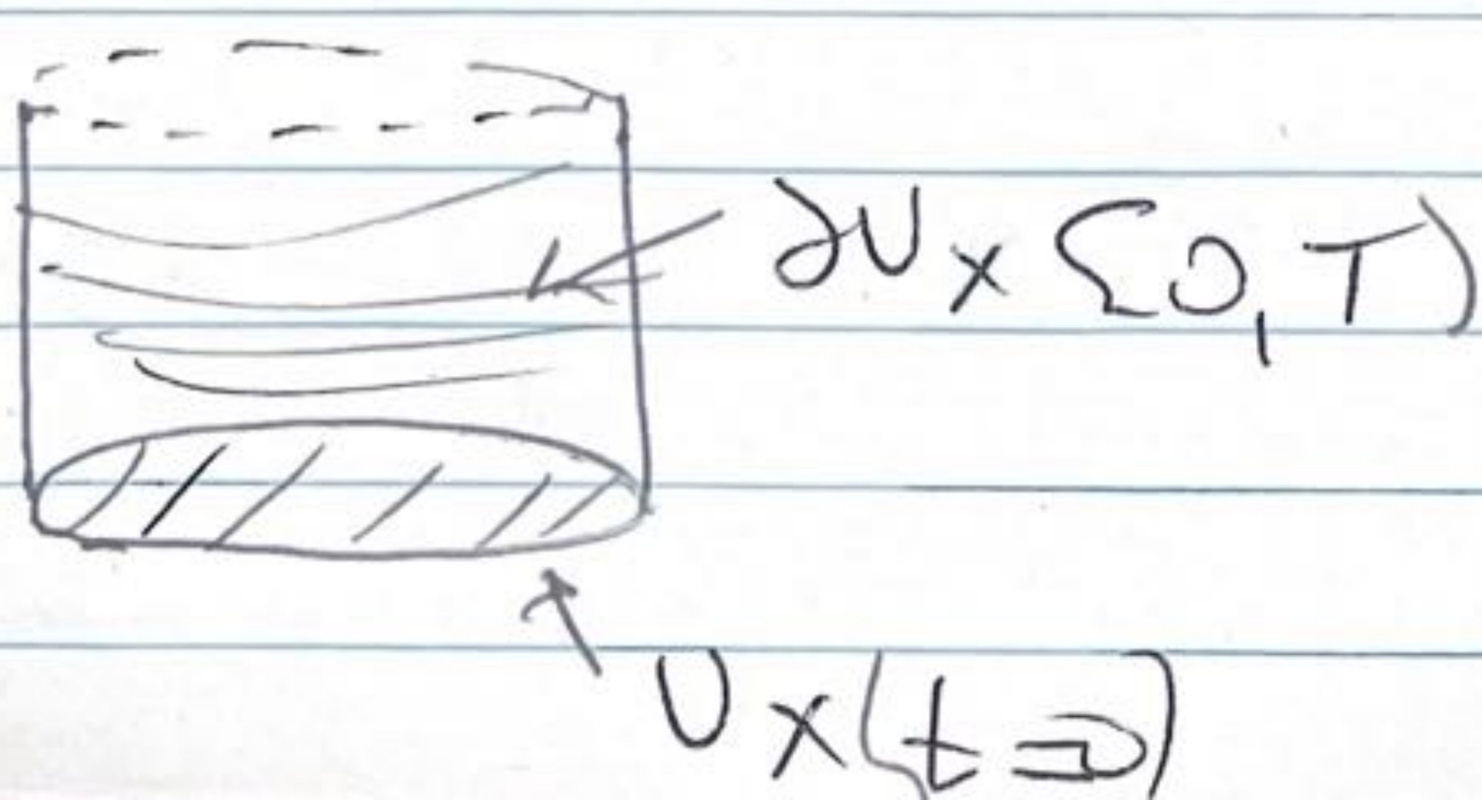
U_T INCLUDES open solid cylinder and top disc But not bottom disc.



<u>PARABOLIC BOUNDARY</u>

$$\Gamma_T = \left(\partial U \times [0, T)\right) \cup \left(U \times \{t = 0\}\right)$$

PICTURE



FUNCTION SPACE

$$C_1^2(U_T) = \left\{ u = u(t,\vec{x}) : u,\ u_t,\ \frac{\partial u}{\partial x_j},\ \frac{\partial^2 u}{\partial x_i \partial x_j} \in C^0(U_T) \quad i,j = 1 \ldots n \right\}$$

THM (MAXIMUM PRINCIPLE)

Let $u \in C_1^2(U_T)$, $u \in C^0(\overline{U_T})$

CTS ON CLOSED CYLINDER

satisfy

$$u_t = k\Delta u \qquad (t,x) \in U_T$$

Then

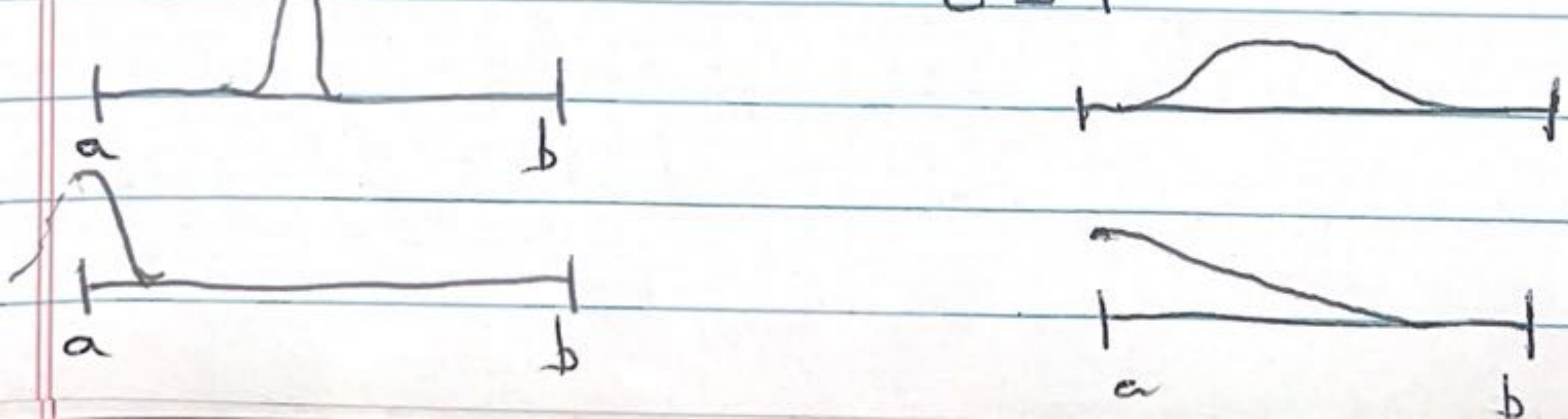
$$\max_{\overline{U_T}} u(t,\vec{x}) = \max_{\Gamma_T} u(t,\vec{x})$$

ie Max value of u either occurs at time zero or on the boundary of U.

INTUITION Heat Diffuses

$t=0$ $\qquad$ $t=T$

OR



INFORMAL PROOF

Suppose u has local max at $(t_0, \vec{x}_0) \in (0,T) \times U$.
ie NOT on Γ_T.
Then

$$\left.\begin{array}{l} u_t(t_0, \vec{x}_0) = 0 \\ \nabla u\,(t_0, \vec{x}_0) = 0 \end{array}\right\} \text{ CRITICAL POINT}$$

$\Delta u \leq 0$ as $\dfrac{\partial^2 u}{\partial x_j^2} \leq 0 \ \forall j$ at $(t_0, \vec{x}_0)$

since slice of u in plane $\{x_i = 0 \ \ \forall i \neq j\}$

has local max

Then by PDE @ $(t_0, \vec{x}_0)$:

$$0 = u_t = k \Delta u \leq 0$$

would give $\times$ if $\Delta u < 0$.

To deal with case of a degenerate max where $\Delta u = 0$ replace u by

$$v(t, x) := u(t, x) + \varepsilon\left(x_1^2 + \cdots + x_n^2\right)$$

which has

$$v_t - k\Delta v = u_t - k\Delta u - 2kn\varepsilon < 0$$

at $(t_0, \vec{x}_0)$. and take $\lim\limits_{\varepsilon \to 0}$.

$\square$