THE DIFFUSION / HEAT EQUATION

Suppose have some quantity, $Q = Q(t, x)$ that depends on time t and position $x \in \mathbb{R}$.

EX Q = Heat Energy or Concentration of Particles.

Let $[Q]$ be units of Q.

LET

$\rho = \rho(t,x)$ = Linear Density of Q UNITS: $\frac{[Q]}{m}$

$\phi = \phi(t,x)$ = Flux of Q UNITS: $\frac{[Q]}{s}$
which measures rate at which Q crosses pt x per unit time.

$S = S(t,x)$ = Rate at which Q is created at (t,x) per unit length, per unit time

UNITS: $\frac{[Q]}{m s}$

Using Conservation of Mass arguement like when we derived the linear transport eq^n get

$$\frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = S \qquad \text{UNITS: } \frac{[Q]}{ms} \qquad (1)$$

IN 3D GET ~~IN 3D GET~~

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\phi} = S \qquad \text{UNITS } \frac{[Q]}{m^3 s} \qquad (1')$$

where $\vec{\phi} = \phi \vec{n}$, ϕ = Rate Q crosses plane thru $\vec{x}$ with normal $\vec{n}$ per unit time UNITS $\frac{[Q]}{m^2 s}$

FICKIAN DIFFUSION

Suppose Flux is proportional to gradient of density:

$$\phi = -D \frac{\partial \rho}{\partial x} \qquad (2)$$

OR IN 3D

$$\vec{\phi} = -D \nabla \rho \qquad (2')$$

Here D = Diffusivity constant UNIT: $\frac{m^2}{s}$

- $D \geq 0$.
- Negative sign in (2) means ρ flows from regions of ↑ density to ↓ density.

PLUG (2) INTO (1):

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\left(D(x) \frac{\partial \rho}{\partial x}\right) = S$$

In case $D(x) = D$ is constant in x got

DIFFUSION EQN

$$\frac{\partial \rho}{\partial t} - D \frac{\partial^2 \rho}{\partial x^2} = S \qquad (3)$$

IN 3D

$$\frac{\partial \rho}{\partial t} - D \Delta \rho = S.$$

In general $D(x)$ = MATRIX depends on x.

Special case $D(x) = D\,Id_{3\times 3}$

HEAT EQN (FOURIER 1822)

Heat and temperature are not same thing.
Temp is relative hotness of coldness of an object measured relative to another object (used as a thermometer)
Heat is a form of energy*

CONNECTION*: To raise/lower temperature of an object need to transfer heat energy to/from that object.

WORK IN 3D

CHOOSE

Q = Heat Energy in J

Then ρ = Heat Energy Density in J/m^3

LET u = Temperature in K

c = Specific Heat Capacity*
= Heat Energy required to raise 1 kg of substance by 1 K. in $\frac{J}{kg\,K}$

δ = Mass Density in $\frac{kg}{m^3}$

Then $\boxed{\rho = \delta c u}$ in J

$$\frac{J}{m^3} = \left(\frac{kg}{m^3}\right)\left(\frac{J}{kg\,K}\right) K$$

$\boxed{\text{FOURIER'S LAW}}$

Heat Energy flows in dir^n of negative temp gradient:

HEAT FLUX $\quad \vec{\phi} = -k\nabla u$

k = THERMAL CONDUCTIVITY IN $\frac{J}{msK}$

UNIT CHECK $\quad \vec{\phi} = -k\nabla u$

$$\frac{J}{m^2 s} = \left(\frac{J}{msK}\right)\left(\frac{K}{m}\right) \quad \checkmark$$

Then (1') becomes (IF k is constant)

$$\frac{\partial e}{\partial t} + \nabla\cdot(-k\nabla u) = S$$

with $\quad e = \rho c u$

So

$$\frac{\partial u}{\partial t} + -\left(\frac{k}{\rho c}\right)\Delta u = \frac{S}{\rho c}$$

SET

$D = \frac{k}{\rho c}$ = THERMAL DIFFUSIVITY, IN $\frac{m^2}{s}$

So get

~~or if k is constant~~

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho c} S \qquad \text{UNITS: } \frac{K}{S}$$

where $D = \frac{k}{\rho c}$ = Thermal Diffusivity UNITS $\frac{m^2}{s}$ ✓✓

BROWNIAN MOTION

- The random motion of large particles that are suspended in a liquid or gas due to collisions with the much smaller liquid/gas particles.

$\boxed{\text{IN 1D}}$

LET $u(t,x)$ = Average # of (large) particles at (t,x) per unit t and x.

Then

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

So u satisfies the diffusion equation.

INTUITIVE DERIVATION

Suppose particles occupy lattice sites

Suppose after time step of size h particles can move left probability p

right p

stay $1-2p$.

So

$$u(t+h, x) = p\,u(t, x-k) + (1-2p)\,u(t, x) + p\,u(t, x+k)$$

OR

$$\frac{u(t+h, x) - u(t, x)}{h} = \left(\frac{k^2 p}{h}\right)\left[\frac{u(t, x+k) - 2u(t, x) + u(t, x-k)}{k^2}\right]$$

Let $D := \frac{k^2 p}{h}$.

Suppose that as $h \to 0$, $k \to 0$ is chosen so that D is constant. $p = p(h, k) = \frac{Dh}{k^2}$ (UNITLESS)

Then get

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

THE FUNDAMENTAL SOLN

PARABOLIC SCALING

Fix scaling parameter $\alpha > 0$.

Let $\tilde{x} = \alpha x, \quad \hat{t} = \alpha^2 t$

Set $v(t,x) := u(\hat{t}, \tilde{x}) = u(\alpha^2 t, \alpha x)$ (4)

Then

$$\frac{\partial v}{\partial t}(t,x) = \alpha^2 \frac{\partial u}{\partial t}(\alpha^2 t, \alpha x)$$

$$\frac{\partial v}{\partial x}(t,x) = \alpha \frac{\partial u}{\partial x}(\alpha^2 t, \alpha x)$$

So if $u_t = D u_{xx}$ — u solves Heat eq^n

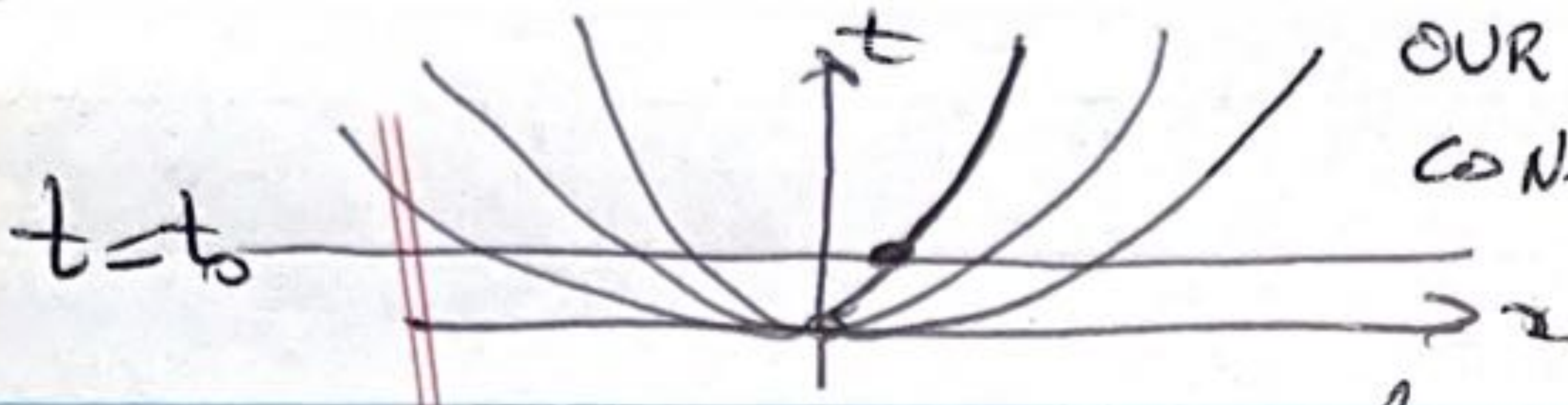
then $v_t = D v_{xx}$ — also solves Heat eq^n

BIG IDEA

Look for solutions $u = u(t,x)$ that do not change when you do a parabolic scaling of (t,x) – space.

ie SATISFY

$$\boxed{u(t,x) = u(\alpha^2 t, \alpha x)} \qquad (5)$$



OUR SELF-SIMILAR SOL^n IS CONST ALONG PARABOLAS

6B

Fix t_0. Suppose know $u(t_0, y)$ $\forall y \in \mathbb{R}$ and know u satisfies parabolic scaling law (S)

Given (t, x) choose α : $t = \alpha^2 t_0$, $\alpha = \sqrt{t/t_0}$

Then

$$u(t,x) = u\left(\alpha^2 t_0, \alpha\left(\frac{x}{\alpha}\right)\right) \quad \text{by def}^n\ \alpha$$

$$\overset{(S)}{=} u\left(t_0, \frac{x}{\alpha}\right)$$

$$= u\left(t_0, \sqrt{t_0}\frac{x}{\sqrt{t}}\right)$$

$$= f(\eta) \qquad \text{where } \eta := \frac{x}{\sqrt{Dt}}$$

is dimensionless.

IF Fix η
u CONSTANT ALONG CURVE
$t = \frac{1}{D\eta^2}x^2$ PARABOLA

So whatever our scale-invariant sol^n is it is really just a function of a single variable, $\eta := \frac{x}{\sqrt{Dt}}$.

Use $u_t = Du_{xx}$ to derive ODE for f.

$$\frac{\partial u}{\partial t} = f'(\eta)\frac{\partial \eta}{\partial t} = f'(\eta)\ -\frac{1}{2}\left(\frac{\eta}{Dt}\right)$$

$$\frac{\partial u}{\partial x} = f'(\eta)\frac{\partial \eta}{\partial x} = f'(\eta)\frac{1}{\sqrt{Dt}} \quad \therefore u_{xx} = \frac{1}{Dt}f''(\eta)$$

So get ODE

$$f''(\eta) = -\tfrac{1}{2}\eta f'(\eta)$$

Let $g(\eta) = f'(\eta)$

$$g' + \tfrac{1}{2}\eta g = 0.$$

Integrating Factor: $e^{\int \frac{1}{2}\eta \, d\eta} = e^{\eta^2/4}$

Get $\frac{d}{d\eta}\left(e^{\eta^2/4} g(\eta)\right) = e^{\eta^2/4}\left[g'(\eta) + \tfrac{1}{2}\eta g(\eta)\right] = 0$

So

$$g(\eta) = C_0 e^{-\eta^2/4}$$

$$f(\eta) = \int_0^\eta C_0 e^{-\eta^2/4} d\eta + C_2$$

$$= 2C_0 \int_0^{\eta/2} e^{-s^2} ds + C_2 \qquad s = \eta/2$$

$$f(\eta) = C_1 \int_0^{\eta/2} e^{-s^2} ds + C_2$$

$$\boxed{u(t,x) = C_1 \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-s^2} ds + C_2}$$

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

$$\boxed{u(t,x) = C_3 \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) + C_2}$$

THE CAUCHY PROBLEM (IVP) FOR HEAT EQ^n

$$\begin{cases} \dfrac{\partial u}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, \quad t \geq 0 \\ u(0, x) = f(x) \end{cases}$$

$\boxed{\text{EXAMPLE}}$

$$f(x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$ HEAVISIDE FUNCTION

Lets use solution

$$u(t, x) = C_1 \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) + C_2$$

to solve this IVP.

$\boxed{\text{CASE } x < 0}$

$$0 = H(x) = \lim_{t \to 0^+} u(t, x)$$

$$= C_1 \lim_{t \to 0^+} \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds + C_2$$

$$= C_1 \int_0^{-\infty} e^{-s^2} ds + C_2$$

$$0 = -C_1 \frac{\sqrt{\pi}}{2} + C_2 \qquad (1)$$

$\boxed{\text{CASE } x > 0}$

$$1 = H(x) = \lim_{t \to 0^+} u(t, x)$$

$$= C_1 \lim_{t \to 0^+} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-s^2} ds + C_2$$

$$1 = C_1 \frac{\sqrt{\pi}}{2} + C_2 \qquad (2)$$

Solving (1) + (2) gives

(1) + (2) : $\quad 2C_2 = 1 \Rightarrow C_2 = \frac{1}{2}$

(1) : $\quad C_1 = \frac{2}{\sqrt{\pi}} C_2 = \frac{1}{\sqrt{\pi}}$

So

$$u(t, x) = \frac{1}{\sqrt{\pi}} \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds + \frac{1}{2}$$

$$\boxed{u(t,x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right]} \qquad (3)$$

NOTE (a) $u(t, 0) = \frac{1}{2} \quad \forall t.$ as $\operatorname{erf}(0) = 0$

(b) $\lim_{t \to \infty} u(t, x) = \frac{1}{2}[1 + \operatorname{erf}(0)] = \frac{1}{2} \quad \forall x$

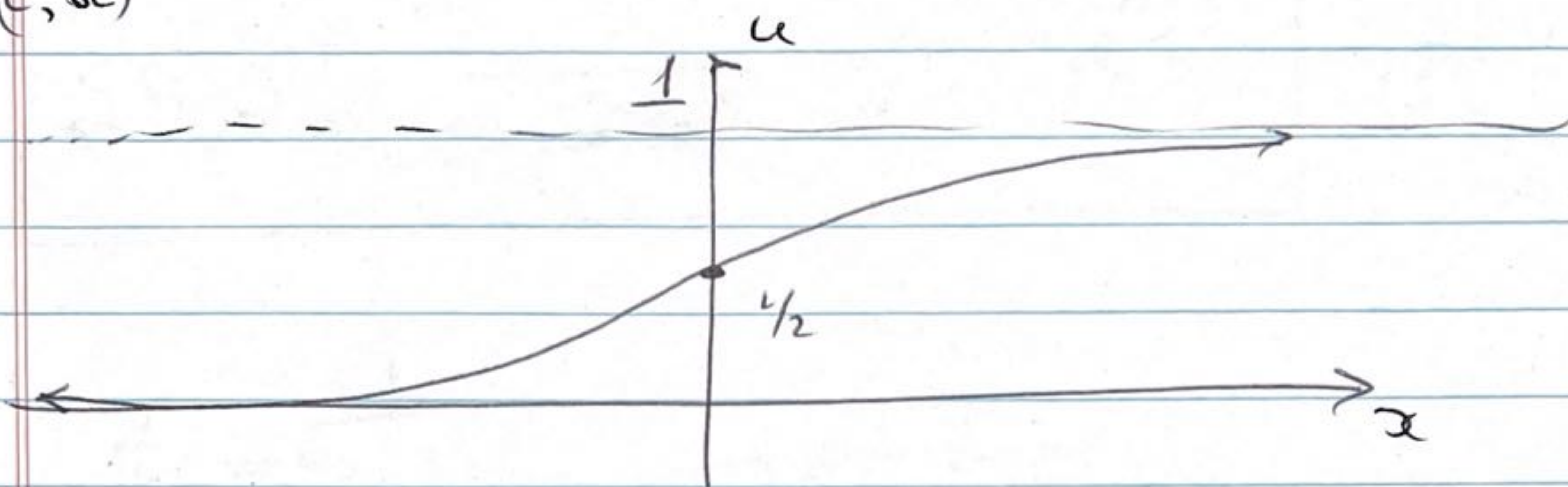
UNIFORM TEMPERATURE

(c) Even though H is not CTS at $x=0$,

$u(t,x)$ is $\underset{C^\infty}{\text{CTS}}$ in x $\quad \underline{\underline{\forall t > 0}}$.

BIG THM: $\boxed{\text{THIS ALWAYS HAPPENS}}$

$u(t,x)$



As $t \uparrow$, solution stretches out horizontally

THE FUNDAMENTAL SOL^n SOLVES

$$\begin{cases} \dfrac{\partial \Omega}{\partial t} = D \dfrac{\partial^2 \Omega}{\partial x^2} & x \in \mathbb{R},\ t > 0 \\ \Omega(0,x) = \delta_0(x) \end{cases} \qquad (4)$$

Since

(a) $H' = \delta_0$

(b) IF u solves $u_t = D u_{xx}$ THEN so does u_x

(11)

The sol^n of (4) is x-derivative of (3)

i.e.
$$\mathcal{U}(t,x) = \frac{\partial}{\partial x} \frac{1}{2}\left[1 + \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds\right]$$

$$\boxed{\mathcal{U}(t,x) = \frac{1}{2\sqrt{\pi D t}} e^{-x^2/4Dt}} \quad (5)$$
$t > 0$

LET $t_n = \frac{1}{4Dn^2}$ $\quad n = 1, 2, 3 \ldots$

Then
$$g_n(x) := \mathcal{U}\left(\frac{1}{4Dn^2}, x\right)$$
$$g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \longrightarrow \delta_0(x)$$

by result from Lecture on Dirac f.

NOTES

(a) Once again $\mathcal{U}(0,x) = \delta_0(x)$ is not even a f^n but for $t>0$, $\mathcal{U}(t,x)$ is C^∞ f^n of x !!

(b) Even though at $t=0$ $\mathcal{U} = 0$ for all $x \in \mathbb{R}$ except $x=0$ for ANY time $t > 0$ (even $t = 10^{-100}$ sec) $\mathcal{U}(t,x) > 0$ $\quad \forall x \in \mathbb{R}$. "HEAT DIFFUSES ∞LY FAST."

(c)

THM

GENERAL CAUCHY PROBLEM FOR HEAT EQ^N

SUPPOSE $f \in C(\mathbb{R})$ IS BOUNDED. Then

~~Let~~ The solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2} & x \in \mathbb{R},\ t > 0 \\ u(0,x) = f(x) \end{cases} \qquad (6)$$

is

$$\boxed{u(t,x) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/4Dt}\, dy} \qquad (7)$$

In particular ① $\lim_{t \to 0^+} u(t,x) = f(x) \quad \forall x$

② $u \in C^\infty((0,\infty) \times \mathbb{R})$

INTUITIVE DERIVATION

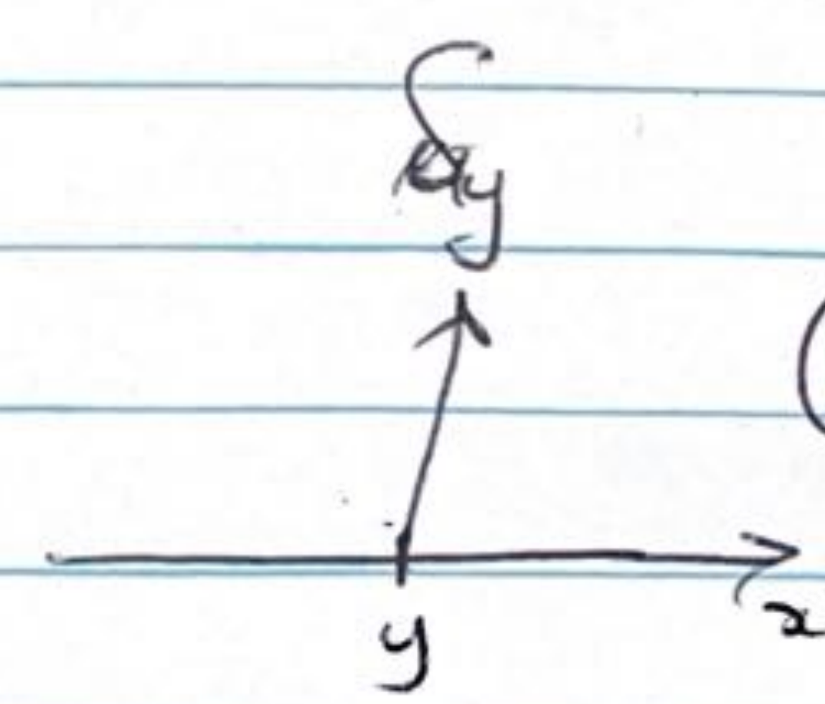
SINCE $S(t,x) = \dfrac{1}{2\sqrt{\pi D t}} e^{-x^2/4Dt}$ solves

$$\begin{cases} u_t = D u_{xx} \\ u(0,x) = \delta_0(x) \end{cases}$$

$$S(t,x-y) = \frac{1}{2\sqrt{\pi D t}} e^{-(x-y)^2/4Dt}$$

solves

$$\begin{cases} u_t = D u_{xx} \\ u(0,x) = \delta_y(x) = \delta_0(x-y) \end{cases} \qquad (8)$$



Also

$$f(x) = \int_{-\infty}^{\infty} f(y)\, \delta_0(x-y)\, dy$$ is a superposition of IC's

So by Principle of Superposition we except 7 to solve 6.

MORE FORMALLY

(a) Assuming we can take $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ inside y-integral:

$$u_t = \int_{\mathbb{R}} f(y) \frac{\partial}{\partial t} S(t, x-y)\, dy$$

$$= \int_{\mathbb{R}} f(y)\, D \frac{\partial^2}{\partial x^2} S(t, x-y)\, dy \quad \text{by } 8$$

$$= D u_{xx}$$

(b) LET $r = \frac{x-y}{2\sqrt{Dt}}$, $y = x - 2\sqrt{Dt}\, r$

C of V.

$$dr = \frac{-dy}{2\sqrt{Dt}}$$

So

$$u(t,x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x - 2\sqrt{Dt}\, r)\, e^{-r^2}\, dr$$

$$\xrightarrow{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x)\, e^{-r^2}\, dr = f(x) \left(\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} dr \right) = f(x).$$

BETTER

Replace everything below ⑦ with: ⑬'

① $Du_t - Du_{xx} = \int (S_t - D S_{xx})(x-y)\, f(y)\, dy = 0$.

② Let $t_n = \frac{1}{4Dn^2} \to 0^+$ as $n \to \infty$.

Set

$$g_n(y) = S(t_n, y) = \frac{n}{\sqrt{\pi}} e^{-n^2 y^2}.$$

We know $L_{g_n} \to \delta_0$ as $n \to \infty$.

So $L_{S_t} \to \delta_0$ as $t \to 0^+$

where $S_t(y) := S(t, y)$.

SHIFTING

$$S_{t,x}(y) := S_t(y-x)$$

So $L_{S_{t,x}} \to \delta_x$ as $t \to 0^+$

NOW $u(t,x) = \int_{-\infty}^{\infty} S_{t,x}(y)\, f(y)\, dy = L_{S_{t,x}}(f)$

So $\lim_{t \to 0^+} u(t,x) = \lim_{t \to 0^+} L_{S_{t,x}}(f) = \delta_x(f) = f(x)$