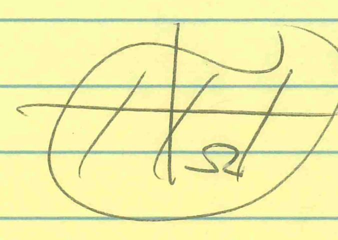
LECTURE 16 $\boxed{\text{GREEN'S FUNCTIONS FOR 1D+2D BVPs}}$

We really want to solve for $u = u(x,y)$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$



For now this is too hard

So instead work in 1D on $[a,b]$, and solve ODE BVP

$$\begin{cases} -u'' = f & a < x < b \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

$\boxed{\text{LIKE } F = ma.}$

FIRST SOLVE IN SPECIAL CASE $f = \delta_\xi$ is UNIT IMPULSIVE FORCE

We call solution to

$$\begin{cases} -u'' = \delta_\xi \quad \text{on } [0,1] \text{ FOR } 0 < \xi < 1 \\ u(0) = 0 = u(1) \end{cases}$$

The Green's Function $u(x) = G_\xi(x) = G(x;\xi)$

So $\begin{cases} -G_\xi'' = \delta_\xi \\ G_\xi(0) = 0 = G_\xi(1) \end{cases}$

We saw before that $f_\xi'' = \delta_\xi$

where $f_\xi(x) = \begin{cases} 0 & \text{IF } x < \xi \\ x-\xi & \text{IF } x > \xi \end{cases}$ is RAMP FN.

<u>Problem</u> IF $0 < \xi < 1$ Then

$$f_\xi(0) = 0 \quad ✓$$

but $f_\xi(1) = 1 - \xi \neq 0.$

So f_ξ does not satisfy BCs.

<u>SOLUTION</u> Write $G_\xi = -f_\xi + z_\xi$ $\qquad z_\xi = z_\xi(x).$

and work out what ODE + BP z_ξ has to solve.

Well $\quad z_\xi = G_\xi + f_\xi$

$$-z_\xi'' = -G_\xi'' - f_\xi'' = \delta_\xi + -\delta_\xi = 0$$

and

$$z_\xi(0) = G_\xi(0) + f_\xi(0) = 0$$

$$z_\xi(1) = G_\xi(1) + f_\xi(1) = 0 + 1 - \xi = 1 - \xi$$

SOLUTION

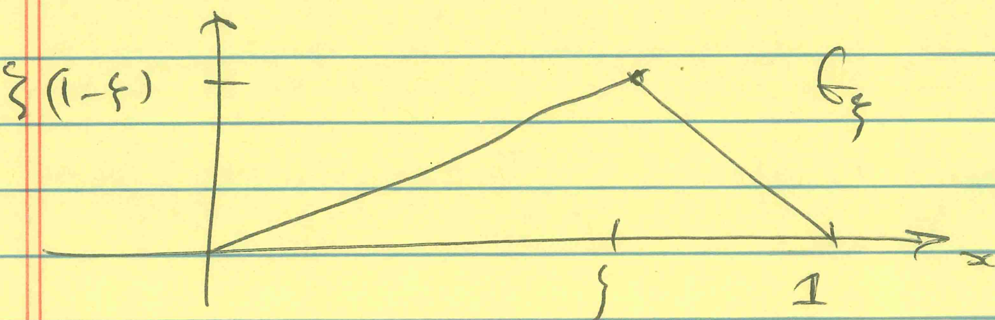
$$z_\xi''(x) = 0 \implies z_\xi(x) = ax+b.$$

$$0 = z_\xi(0) = b \implies z_\xi(x) = ax$$

$$1-\xi = z_\xi(1) = a \implies z_\xi(x) = (1-\xi)x$$

So

$$G_\xi(x) = (1-\xi)x - f_\xi(x) = \begin{cases} (1-\xi)x & \text{IF } x<\xi \\ \xi(1-x) & \text{IF } x>\xi \end{cases}$$



SECOND Use superposition to solve for general forcing function f.

INTUITION
$$f(x) = \delta_x[f] = \int_0^1 \delta_x(\xi)\, f(\xi)\, d(\xi)$$
$$= \int_0^1 \delta(\xi - x)\, f(\xi)\, d\xi$$
$$= \int_0^1 \delta(x-\xi)\, f(\xi)\, d\xi$$

$$\boxed{f(x) = \int_0^1 \delta_\xi(x)\, f(\xi)\, d\xi}$$

Since
$$-G_\xi''(x) = \delta_\xi(x)$$
$$G_\xi(0) = 0 = G_\xi(1)$$

this suggests we define
$$u(x) = \int_0^1 G_\xi(x) f(\xi)\, d\xi$$
$$\boxed{u(x) = \int_0^1 G(x;\xi) f(\xi)\, d\xi}$$

<u>INFORMALLY</u>
$$-u''(x) = -\int_0^1 G_\xi''(x) f(\xi)\, d\xi$$
$$= \int_0^1 \delta_\xi(x) f(\xi)\, d\xi = f(x)$$

and $u(0) = \int_0^1 G_\xi(0) f(\xi)\, d\xi = \int_0^1 0\, d\xi = 0$
$u(1) = 0$ too ✓

<u>FORMALLY</u> from formula for G_ξ:
$$u(x) = \int_0^x \xi(1-x) f(\xi)\, d\xi + \int_x^1 (1-\xi) x f(\xi)\, d\xi$$
$$\boxed{u(x) = (1-x)\int_0^x \xi f(\xi)\, d\xi + x\int_x^1 (1-\xi) f(\xi)\, d\xi}$$
or $u(x) = (1-x)\int_0^x \xi f(\xi)\, d\xi - x\int_1^x (1-\xi) f(\xi)\, d\xi$ ⊛

<u>CHECK</u> $u(0) = 0 = u(1)$ ✓

And

By FTC on (*)

$$u'(x) = -\int_0^x \xi f(\xi)\,d\xi + (1-x)x\,f(x) - \int_1^x (1-\xi) f(\xi)\,d\xi - x(1-x)\,f(x)$$

$$u'(x) = -\int_0^x \xi f(\xi)\,d\xi - \int_1^x (1-\xi) f(\xi)\,d\xi$$

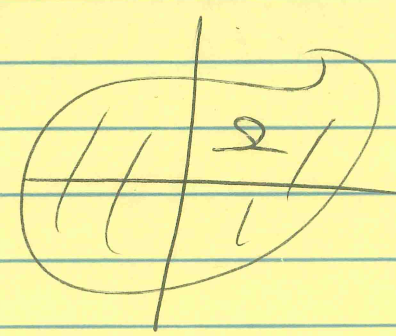
So

$$u''(x) = -x f(x) - (1-x) f(x) = -f(x)$$

or $\quad -u'' = f$ ✓

$\boxed{\text{2 D CASE}}$

<u>GOAL</u> SOLVE $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$

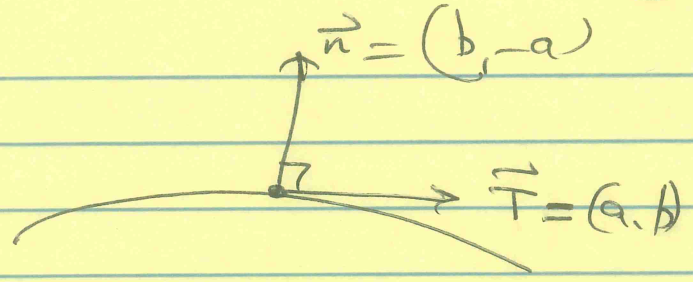


<u>PRELIMINARIES:</u> <u>GREEN'S IDENTITIES</u> (ANALOGS OF INTEGR^{N} BY PARTS)

<u>GREEN'S THM</u> Let $P = P(x,y)$, $Q = Q(x,y)$

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial\Omega} P\,dx + Q\,dy.$$

SET $\vec{F} = Q\vec{i} - P\vec{j}$.



Then

$$\nabla \cdot \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and

$$\int_{\partial\Omega} P\,dx + Q\,dy = \int_{\partial\Omega} (P, Q) \cdot \vec{T}\,ds$$

$$= \int_{\partial\Omega} (aP + bQ)\,ds$$

$$= \int_{\partial\Omega} (\vec{F} \cdot \vec{n})\,ds$$

So get

DIVERGENCE FORM OF GREEN'S THM

$$\boxed{\iint_{\Omega} \nabla \cdot \vec{F}\,dA = \int_{\partial\Omega} (\vec{F} \cdot \vec{n})\,ds} \qquad (1)$$

SET $\vec{F} = u\vec{v}$ $\qquad$ $u = u(\vec{x})$, FUNCTION $\qquad$ $\vec{v} = \vec{v}(\vec{x})$ VECTOR FIELD

Then

$$\nabla \cdot (u\vec{v}) = \nabla u \cdot \vec{v} + u\,\nabla \cdot \vec{v}$$ PRODUCT RULE (2)

PLUG INTO (1):

$$\iint_{\Omega} (\nabla u \cdot \vec{v} + u\,\nabla \cdot \vec{v})\,dA = \int_{\partial\Omega} u(\vec{v} \cdot \vec{n})\,ds.$$ INTEGRATION BY PARTS.

SET $\vec{v} = \nabla v$ where $v = v(\vec{x})$ is a FUNCTION and notice $\nabla \cdot \vec{v} = \nabla \cdot \nabla v = \Delta v$.

So get

$$\iint_\Omega (u \Delta v + \nabla u \cdot \nabla v)\, dA = \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}}\, ds \qquad (3)$$

SWITCH u, v IN (3) to get (3'). Subtract (3'), (3) to get

$$\boxed{\iint_\Omega (u \Delta v - v \Delta u)\, dA = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right) ds} \qquad (4)$$

– Green's 2nd Identity.

FUNDAMENTAL SOLN TO LAPLACE'S EQN

THM Let $F_{\vec{\xi}}(\vec{x}) = -\frac{1}{2\pi} \log |\vec{x} - \vec{\xi}|$

where $\vec{x} = (x, y)$, $\vec{\xi} = (\xi, \eta)$

Then

$$\boxed{-\Delta F_{\vec{\xi}} = \delta_{\vec{\xi}}}$$

ANALOGUE OF RAMP FUNCTION

Pf Idea

In polar coords (thinking $\vec{\xi}$ is at origin)

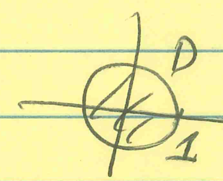
$$F = -\frac{1}{2\pi}\log(r).$$

We saw B4 that $\Delta(\log r) = 0$ for $r \neq 0$.

So we guess $\exists C$:

$$+\Delta \log(r) = C\,\delta_{\vec{0}}.$$

By Property of $\delta_{\vec{0}}$:



$$C = \iint_D C\,\delta_{\vec{0}}(\vec{x})\,dA$$

$$C = +\iint_D \Delta(\log(r))\,dA$$

Apply Green's 2nd Identity 4 with $u = +1$, $v = \log r$ to get

$$C = +\iint_D \Delta(\log r)\,dA$$

$$= \int_{\partial D} (+1)\,\frac{\partial}{\partial \vec{n}}(\log r)\,ds \qquad \text{as } \Delta(+1) = 0, \ \frac{\partial}{\partial \vec{n}}(+1) = 0.$$

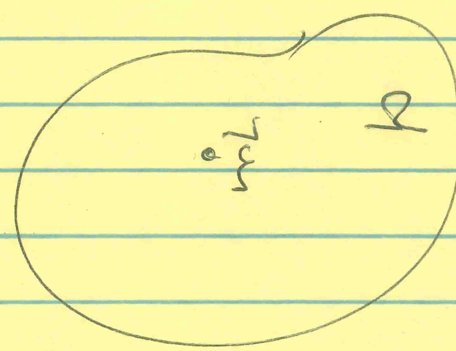
$$= +\int_{\partial D} \frac{\partial}{\partial r}(\log r)\,ds$$

$$= + \int_{\partial D} \frac{1}{r}\, ds \equiv + \int_{\partial D} 1\, ds = 2\pi$$

So $$\Delta\left(\frac{1}{2\pi}\log r\right) = \delta_{\vec{0}}$$

Hence $$-\Delta F_{\vec{\xi}} = \delta_{\vec{\xi}}.$$

Next Fix $\vec{\xi} \in \Omega$.



SOLVE
$$\begin{cases} -\Delta u = \delta_{\vec{\xi}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

CALL THE SOLN $u(\vec{x}) = G_{\vec{\xi}}(\vec{x}) = G(\vec{x};\vec{\xi})$.

As before we must adjust Fundamental Sol^n to get correct boundary values:

Write $$G_{\vec{\xi}} = F_{\vec{\xi}} + z_{\vec{\xi}}.$$

Then $$\Delta z_{\vec{\xi}} = \Delta G_{\vec{\xi}} - \Delta F_{\vec{\xi}} = -(\delta_{\vec{\xi}} - \delta_{\vec{\xi}}) = 0$$

and on $\partial\Omega$ $$z_{\vec{\xi}} = 0 - F_{\vec{\xi}}$$

SUPPOSE we can solve

$$\begin{cases} \Delta Z_{\vec{\xi}} = 0 & \text{IN } \Omega \\ Z_{\vec{\xi}}(\vec{x}) = \frac{1}{2\pi} \log |\vec{x} - \vec{\xi}| & \text{on } \partial\Omega \end{cases} \quad (5)$$

Then

$$G_{\vec{\xi}} = F_{\vec{\xi}} + Z_{\vec{\xi}}$$ is our Green's Fn for Ω.

FINALLY TO SOLVE

$$\begin{cases} -\Delta u = f & \text{IN } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

Set $v = G_{\vec{\xi}}$, $u = u$ in (4) to get

$$\iint_{\Omega} (u \, \Delta G_{\vec{\xi}} - G_{\vec{\xi}} \Delta u) \, dA = \int_{\partial\Omega} \left(u \frac{\partial G_{\vec{\xi}}}{\partial \vec{n}} - G_{\vec{\xi}} \frac{\partial u}{\partial \vec{n}} \right) ds$$

Since $\Delta G_{\vec{\xi}} = -\delta_{\vec{\xi}}$, $G_{\vec{\xi}} = 0$ on $\partial\Omega$

$\Delta u = f$ in Ω, $u = h$ on $\partial\Omega$

we get

$$\cancel{u(\vec{x}) = \iint_{\Omega} \delta_{\vec{\xi}}(\vec{x}) u(\vec{\xi}) \, dA = -\iint_{\Omega} u \, \Delta G_{\vec{\xi}} \, dA}$$

Suppose we can solve

$$\begin{cases} \Delta z_{\vec{\xi}} = 0 & \text{in } \Omega \\ z_{\vec{\xi}}(\vec{x}) = \frac{1}{2\pi} \log|\vec{x}-\vec{\xi}| & \text{on } \partial\Omega \end{cases}$$

C^∞ on $\partial\Omega$ as $\vec{\xi} \notin \partial\Omega$

For ex

* If Ω = RECTANGLE or DISC can do This using Fourier Series

Then Green's Function for Δ on Ω is

$$G_{\vec{\xi}} = F_{\vec{\xi}} + z_{\vec{\xi}}$$

$$G(\vec{x};\vec{\xi}) = F(\vec{x};\vec{\xi}) + z(\vec{x};\vec{\xi})$$

SYMMETRY By construction

$$G(\vec{x};\vec{\xi}) = G(\vec{\xi};\vec{x})$$

So

$$\Delta_{\vec{\xi}} G(\vec{x};\vec{\xi}) = \Delta_{\vec{\xi}} G(\vec{\xi};\vec{x})$$

$$= \Delta_{\vec{x}} G(\vec{x};\vec{\xi})$$

$$= -\delta_{\vec{\xi}}(\vec{x}) = -\delta(\vec{x}-\vec{\xi}) = -\delta_{\vec{x}}(\vec{\xi})$$

So

$$u(\vec{x}) = \iint_{\Omega} u(\vec{\xi})\, \delta_{\vec{x}}(\vec{\xi})\, dA(\xi)$$

$$= -\iint_{\Omega} u(\vec{\xi})\, \Delta_{\xi} G(\vec{x};\vec{\xi})\, dA(\xi)$$

$$\overset{4}{=} -\iint_{\Omega} G(\vec{x};\vec{\xi})\, \overset{-f}{\overset{\|}{\Delta u}}\, dA - \int_{\partial\Omega} \Big(\underset{h}{\underset{\|}{u}} \frac{\partial G}{\partial n_{\xi}} - \underset{0}{\underset{\|}{G}} \frac{\partial u}{\partial n_{\xi}} \Big) ds$$

So <u>GREEN'S REPN THM</u>

$$\boxed{u(\vec{x}) = \iint_{\Omega} G(\vec{x};\vec{\xi})\, f(\vec{\xi})\, dA - \int_{\partial\Omega} \frac{\partial G(\vec{x};\xi)}{\partial \vec{n}_{\xi}}\, h(\xi)\, ds}$$

Solves $\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$

IN SPECIAL CASE Ω = DISC, $f = 0$ recover POINCARE'S FORMULA.