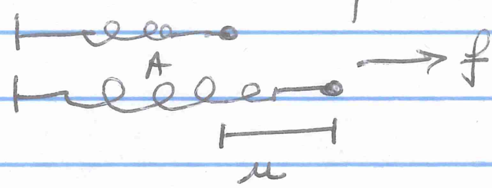
LECTURE 15 — GENERALIZED FUNCTIONS — [OLVER 6.1]

ANALOGY FROM LINEAR ALGEBRA

Consider a simple model for system of masses and springs.

For a single mass, single spring



Force = (Spring Constant) x Displacement

$$f = Au$$

For a system of n masses

$$\vec{f} = A\vec{u}$$

where $\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathbb{R}^n$, f_j = FORCE on j^{th} mass

$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ u_j = Displacement of j^{th} mass

A_{ij} = Constant of Spring connecting m_j and m_i

PROBE THE SYSTEM using IMPULSIVE FORCE

$$\vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$ models unit force applied solely to j-th mass

The force $\vec{e}_j$ results in displacements u_{jk} of all masses m_k.

Write $\vec{u}_j = \begin{pmatrix} u_{j1} \\ \vdots \\ u_{jn} \end{pmatrix} \in \mathbb{R}^n$.

We can solve $A\vec{u}_j = \vec{e}_j$ for $\vec{u}_j$

SINCE $\{\vec{e}_1, \ldots \vec{e}_n\}$ is standard basis for $\mathbb{R}^n$

For any $\vec{f}$: $\vec{f} = \sum_{j=1}^{n} f_j \vec{e}_j$

So sol^n of

$$A\vec{u} = \vec{f} \quad \text{is}$$

$$\boxed{\vec{u} = \sum_{j=1}^{n} f_j \vec{u}_j}$$

as

$$A\vec{u} = A\left(\sum_{j=1}^{n} f_j \vec{u}_j\right) \overset{\text{LIN}}{=} \sum_{j=1}^{n} f_j A\vec{u}_j = \sum_{j=1}^{n} f_j \vec{e}_j = \vec{f}.$$

GOAL EXTEND IDEA to DE's

SIMPLEST CASE Find $u = u(x)$ on $0 \le x \le 1$ so that

$$\begin{cases} -cu'' = f \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

ODE

BVP

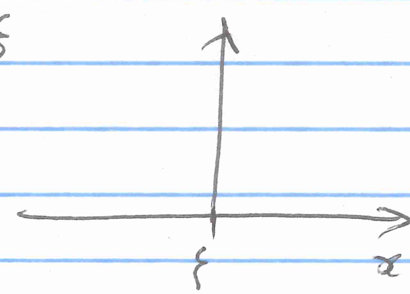
THE DELTA FUNCTION

Analogue of unit ~~impluse~~ impulse $\vec{e}_j$ is DIRAC DELTA FUNCTION, $\delta_\xi(x)$, which should model an impulsive force concentrated solely at $x = \xi$.

So want

ⓐ $\delta_\xi(x) = 0 \quad \forall x \neq \xi$

ⓑ $\int_{-\infty}^{\infty} \delta_\xi(x)\, dx = 1$



ⓐ $\iff (\vec{e}_j)_k = 0 \quad \forall k \neq j$

ⓑ $\iff \sum_{k=1}^{n} (\vec{e}_j)_k = 1$ TOTAL FORCE IS 1

PROBLEM IMPOSSIBLE FOR ⓐ, ⓑ BOTH TO HOLD ~~FOR~~ IN STANDARD CALCULUS

2 APPROACHES

1 VIA LIMITS

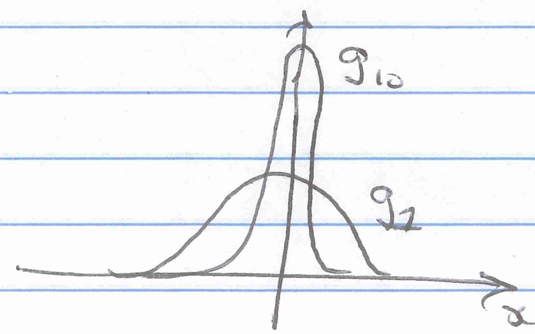
Regard $\delta_\xi = \lim_{n\to\infty} g_n$

where $g_n : \mathbb{R} \to \mathbb{R}$ are smooth functions satisfying

(a) $g_n(x) \to 0$ as $n \to \infty$ $\forall x \neq \xi$

(b) $\int_{-\infty}^{\infty} g_n(x)\,dx = 1.$

So as $n \to \infty$ g_n behaves more + more like impulsive force.



EX

$$g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

has

$$\delta_0 = \lim_{n\to\infty} g_n \quad \text{and} \quad \delta_\xi(x) = \lim_{n\to\infty} g_n(x-\xi)$$

CHECK

(a) IF $x \neq 0$ Then

$$\lim_{n\to\infty} g_n(x) = \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \frac{n}{e^{n^2x^2}}$$

$$\stackrel{\text{L'HOP}}{=} \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \frac{1}{2nx^2 e^{n^2x^2}} = 0$$

(a') IF $x = 0$ Then $g_n(0) = \frac{n}{\sqrt{\pi}} \to \infty$ as $n \to \infty$

(b) $$\int_{-\infty}^{\infty} g_n(x)\,dx = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2 x^2}\,dx \qquad \begin{array}{l} u = nx \\ du = n\,dx \end{array}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du = 1$$

ASIDE

$$1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\,dx$$

$$\neq \underbrace{\int_{-\infty}^{\infty} \lim_{n\to\infty} g_n(x)\,dx = \int_{-\infty}^{\infty} 0\,dx = 0}$$

as $\lim_{n\to\infty} g_n(x) = 0$ everywhere except at single point $x = 0$.

So cannot always take limits inside integrals!

$\boxed{\text{(2) VIA DUALITY}}$

$\boxed{\text{SOME LINEAR ALGEBRA}}$

DEF A <u>LINEAR FUNCTIONAL</u> or <u>DISTRIBUTION</u> on a vector space V is a linear transformation

$$L : V \to \mathbb{R}$$

THM (CASE $V = \mathbb{R}^n$)

For every linear functional $L : \mathbb{R}^n \to \mathbb{R}$

$$\exists \vec{a} \in \mathbb{R}^n : \quad L(\vec{x}) = \vec{x} \cdot \vec{a} \qquad \forall \vec{x} \in \mathbb{R}^n.$$

PF

Set $a_j = L(\vec{e}_j)$, $\quad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

Then since $\vec{x} = \sum_{j=1}^{n} x_j \vec{e}_j$ we have

$$L(\vec{x}) = \sum_{j=1}^{n} x_j L(\vec{e}_j) = \sum_{j=1}^{n} x_j a_j = \vec{x} \cdot \vec{a} \quad \square$$

When studying ODE's we often work with the ∞-DIM^L VS

$$C^0[a,b] = \{ u : [a,b] \to \mathbb{R} \,/\, u \text{ is CTS} \}$$

with L^2-INNER PRODUCT

$$\langle u, v \rangle = \int_a^b u(x) v(x) \, dx$$

DEF For any $g \in C^0[a,b]$ we define a LINEAR FUNCTIONAL, $L_g : C^0[a,b] \to \mathbb{R}$ by

$$L_g(u) = \langle u, g \rangle = \int_a^b u(x) g(x) \, dx$$

INTUITIVELY we would like

$$\int_a^b \delta_\xi(x)\, u(x)\, dx = u(\xi) \qquad \text{for } \xi \in (a,b)$$

But since integrand is ZERO everywhere except $x = \xi$ this is nonsense.

Instead we make sense of δ_ξ by defining it to be a linear functional on $C^0[a,b]$, i.e. a DISTRIBUTION:

DEF Let $\xi \in (a,b)$. Define

$$\delta_\xi : C^0[a,b] \longrightarrow \mathbb{R} \qquad \text{by}$$

$$\boxed{\delta_\xi[u] = u(\xi)}$$

NOTE

$$\begin{aligned} \delta_\xi(a_1 u_1 + u_2) &= (a_1 u_1 + u_2)(\xi) \\ &= a_1 u_1(\xi) + u_2(\xi) \\ &= a_1 \delta_\xi[u_1] + \delta_\xi[u_2] \end{aligned}$$

So δ_ξ is linear!!

~~DEF Let~~ $\cancel{\{L_n\}_{n=1}^{\infty}}$ ~~be a sequence of distributions,~~

$$\cancel{L_n : C^0[a,b] \longrightarrow \mathbb{R}.}$$

~~We say~~ $\cancel{L_n \longrightarrow L}$ ~~i~~

TERMINOLOGY We call the functions u upon which the distributions act, test functions.

DEF Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of distributions and L a distribution, ie

$$L_n, L : C^0[a,b] \longrightarrow \mathbb{R} \text{ are linear.}$$

We say $L_n \to L$ ~~if~~ POINTWISE if

$$L_n[u] \xrightarrow[\text{IN } \mathbb{R}]{} L[u] \quad \forall u \in C^0[a,b]$$

THM Let $g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \in C^0(\mathbb{R})$.

Then $L_{g_n} \longrightarrow \delta_0$ pointwise as distributions on $C^0(\mathbb{R})$.

PF Let $u \in C^0(\mathbb{R})$. ~~Consider~~

$\boxed{\text{FOR TECHNICAL REASONS WE ASSUME } u \text{ IS BOUNDED } |u(x)| \leq M}$

$$\lim_{n\to\infty} L_{g_n}[u] = \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)\, u(x)\, dx$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2x^2} u(x)\, n\, dx$$

LET $y = nx$, $dy = n\,dx$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} u\left(\frac{y}{n}\right) dy$$

$$\overset{*}{=} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \lim_{n\to\infty} \left[e^{-y^2} u\left(\frac{y}{n}\right)\right] dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} u(0)\, dy$$

$$= u(0) \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy \right) = u(0) = \delta_0[u] \quad \blacksquare$$

JUSTIFICATION FOR (*)

We can take limit inside integral as

$$F_n(y) = e^{-y^2} u\left(\frac{y}{n}\right)$$

has property

$$F_n(y) \xrightarrow{n \to \infty} e^{-y^2} u(0) \quad \text{POINTWISE}$$

as u is CB

and

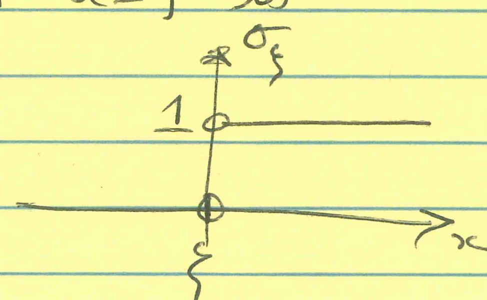
$$|F_n(y)| \le M e^{-y^2}$$ is bounded by a function with a finite integral. $\blacksquare$

– See Lebesgue Dominated Convergence Theorem in Measure Theory $\square$

INTEGRATION + DIFFERENTIATION OF δ_ξ

<u>Def</u> The UNIT STEP FUNCTION AT $x = \xi$ is

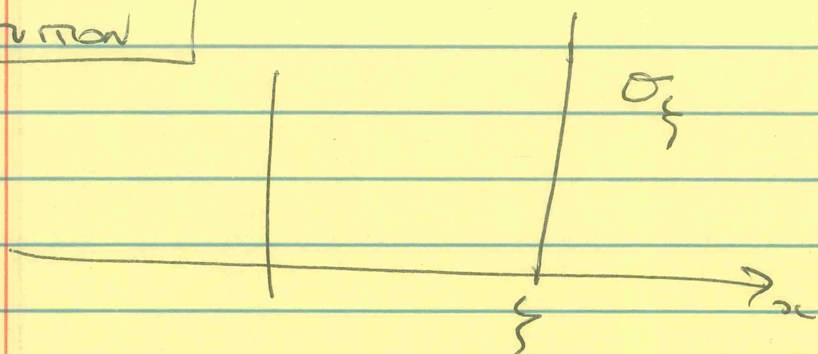
$$\sigma_\xi(x) = \begin{cases} 0 & \text{IF } x < \xi \\ 1 & \text{IF } x > \xi \end{cases}$$



<u>CLAIM</u> $$\int_a^x \delta_\xi(t)\,dt = \sigma_\xi(x)$$

<u>PF</u>

1 $\boxed{\text{INTUITION}}$



- IF $x < \xi$ Then $\sigma_\xi(t) = 0 \quad \forall\, a < t < x$ So $\int_a^x \delta_\xi(t)\,dt = 0$

- IF $x > \xi$ Then intuitively we have

$$\int_a^x \delta_\xi(t)\,dt = 1.$$

2 $\boxed{\text{VIA LIMITS}}$ $\quad \xi = 0.$

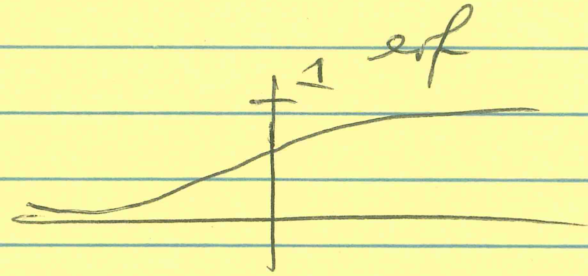
$a = -\infty$ is OK

We expect $$\int_{-\infty}^x \delta_0(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^x g_n(t)\,dt$$

$$= \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \int_{-\infty}^x e^{-n^2t^2}\, n\,dt$$

$$= \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \int_{-\infty}^{nx} e^{-s^2}\,ds$$

$$= \lim_{n\to\infty} \operatorname{erf}(nx)$$



where

$$\operatorname{erf}(x) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^x e^{-s^2}\,ds$$

IF $x < 0$ Then $\mathrm{erf}(nx) \longrightarrow \mathrm{erf}(-\infty) = 0$ as $n \to \infty$

IF $x > 0$ Then $\mathrm{erf}(nx) \longrightarrow \mathrm{erf}(+\infty) = 1$ as $n \to \infty$.

So

$$\int_{-\infty}^{x} \delta_0(t)\,dt = \lim_{n\to\infty} \mathrm{erf}(nx) = \sigma_0(x).$$

$\boxed{\text{DIFFERENTIATING DISTRIBUTIONS}}$

For simplicity we suppose our test functions satisfy

- $u \in C^{\infty}(\mathbb{R})$
- $\int_{-\infty}^{\infty} |u(x)|\,dx < \infty$.

SAY $\boxed{u \in C_0^{\infty}(\mathbb{R})}$

$\therefore$ $u(x) \to 0$ as $x \to \pm\infty$.

THM

SUPPOSE g is differentiable. Then

$$\boxed{L_{g'}[u] = -L_g[u']}$$

INTEGRATION BY PARTS.

PF

$$L_{g'}[u] = \int_{-\infty}^{\infty} g'(x)u(x)\,dx$$

$$= \Big[g(x)u(x)\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} g(x)u'(x)\,dx$$

$$= 0 - L_g[u'].$$ as $u(\pm\infty) = 0$.

This suggests

DEF Let $L : C_0^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$ be a distribution

Define

$$L'[u] := -L[u']$$

EXS

1. If g is differentiable

$$(L_g)'[u] \overset{\text{DEF}}{=} -L_g[u'] \overset{\text{THM}}{=} L_{g'}[u]$$

So

$$(L_g)' = L_{g'}$$

If our distribution is integration against g

Then derivative of L_g is just integration against g'.

2. CLAIM $\boxed{\sigma_\xi' = \delta_\xi}$

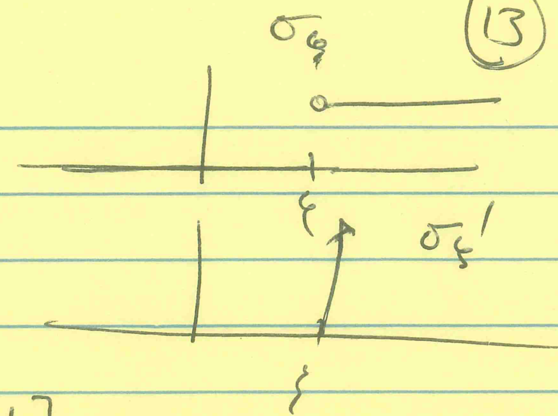
PF

$$\cancel{\sigma_\xi'[u] \quad L_\delta}$$

$$\sigma_\xi[u] = L_{\sigma_\xi}[u] = \int_{-\infty}^{\infty} \sigma_\xi(x)\, u(x)\, dx.$$

defines σ_ξ as a distribution ~~even Th~~

BUT σ_ξ is NOT differentiable! So 1 does not apply.



INSTEAD

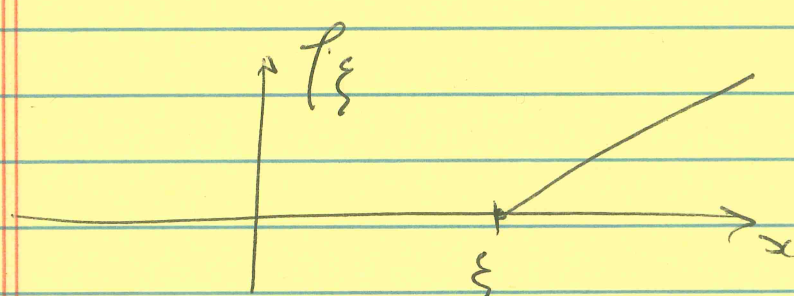
$$\sigma_\xi'[u] = \left(L_{\sigma_\xi}\right)'[u]$$
$$\stackrel{\text{DEF}}{=} -L_{\sigma_\xi}[u']$$
$$= -\int_{-\infty}^{\infty} \sigma_\xi(x)\, u'(x)\, dx$$
$$= -\int_{\xi}^{\infty} u'(x)\, dx$$
$$= -[u(\infty) - u(\xi)] \qquad \text{FTC}$$
$$= u(\xi) = \delta_\xi[u].$$

So $\quad \sigma_\xi' = \delta_\xi \quad$ holds

$\boxed{\text{RAMP FUNCTION}}$

DEF

Define $\quad \rho_\xi(x) = \begin{cases} 0 & \text{IF } x < \xi \\ x-\xi & \text{IF } x > \xi \end{cases}$



RAMP FUNCTION

CLAIM

① $(\rho_\xi)' = \sigma_\xi$ as distributions

② So $(\rho_\xi)'' = \delta_\xi$.

③ ALT $\int_{-\infty}^{x} \sigma_\xi(t)\,dt = \rho_\xi(x)$.

PF

① $(\rho_\xi)'[u] = -\rho_\xi[u']$

$$= -\int_{-\infty}^{\infty} \rho_\xi(x)\, u'(x)\,dx$$

$$= -\int_{\xi}^{\infty} (x-\xi)\, u'(x)\,dx$$

$U = x-\xi$
$V' = u'$
$U' = 1$
$V = u$

PARTS $= -\Big[(x-\xi)\, u(x)\Big]_{\xi}^{\infty} + \int_{\xi}^{\infty} 1 . u(x)\,dx$

$$= 0 + \int_{\xi}^{\infty} u(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sigma_\xi(x)\, u(x)\,dx$$

$$= \sigma_\xi[u]$$

✓

□