$$+ \frac{\cos\theta \sin\theta}{r^2} \frac{\partial^2 u_{pc}}{\partial\theta} + \frac{\sin^2\theta}{r^2} \frac{\partial^2 u_{pc}}{\partial\theta^2}$$

Similarly for $\frac{\partial^2 u_{pc}}{\partial y^2}$. Sum to get

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\boxed{\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}}$$

ALSO BC is $\boxed{u(1,\theta) = h(\theta)}$

search for separable sol^n

$$\boxed{u(r,\theta) = v(r)\, w(\theta)}$$

Get

$$0 = \Delta u = v'' w + \frac{1}{r} v' w + \frac{1}{r^2} w''$$

$$0 = (r^2 v'' + r v') w + w''$$

OR

$$\frac{r^2 v'' + r v'}{v} = -\frac{w''}{w} = \lambda.$$

Gives

$$\boxed{\begin{array}{l} r^2 v'' + r v' - \lambda v = 0 \\ \\ w'' + \lambda w = 0 \end{array}}$$. $w(0) = w(2\pi)$

As before get

$$\left.\begin{array}{l} w_0(\theta) = 1 \\ w_n(\theta) = \cos(n\theta) \\ \tilde{w}_n(\theta) = \sin(n\theta) \end{array}\right\} n = 1, 2, \ldots$$

and $\boxed{\lambda_n = n^2}$

Then ODE for v is

$$\boxed{r^2 v'' + r v' - n^2 v = 0} \qquad v = v(r)$$

GUESS

$$v(r) = r^k$$

Get

$$k(k-1) r^k + k r^k - n^2 r^k = 0$$

$$\Rightarrow \quad k^2 = n^2$$

$$\Rightarrow \quad k = \pm n$$

$$v_1(r) = r^n \qquad v_2(r) = r^{-n}. \qquad n = 1, 2, \ldots$$

NOTE $n=0$ yields only 1 sol^n $V_1(r)=1$.

Better for $n=0$ have

$$r^2V'' + rV' = 0$$

Let $f = V'$

Then
$$rf' + f = 0$$

$$\frac{f'}{f} = -\frac{1}{r}$$

$$f = -A\log r$$

So get LI sol^{ns}

$$\boxed{V_1(r) = 1, \quad V_2(r) = \log r}$$

So get $u = u(r,\theta)$ as

1	$r^n\cos(n\theta)$	$r^n\sin(n\theta)$	← CTS at $r=0$ KEEP
$\log r$	$r^{-n}\cos(n\theta)$	$r^{-n}\sin(n\theta)$	← NOT CTS at $r=0$. THROW AWAY

UPSHOT

$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)} \quad (A)$$

and

$$h(\theta) = u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$$

gives

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\phi)\cos(n\phi)\, d\phi \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\phi)\sin(n\phi)\, d\phi \end{aligned}} \quad (B)$$

as sol^n of BVP for Laplace Eqn

COOL CALCULATION: We can sum series for u to get

THM $\boxed{\text{POISSON INTEGRAL FORMULA}}$

The sol^n to
$$\Delta u = 0 \text{ in } D$$
$$u = h \text{ on } \partial D$$
is

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\, K(r, \theta-\phi)\, d\phi$$

where

$$K(r,\theta) = \frac{1-r^2}{1+r^2-2r\cos\phi} \qquad \text{POISSON KERNEL}$$

PF

PLUG (B) into (A)

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\, d\phi$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} h(\phi)\left[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right] d\phi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} h(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos[n(\theta-\phi)]\right] d\phi$$

NOW $\qquad z = re^{i\theta}$

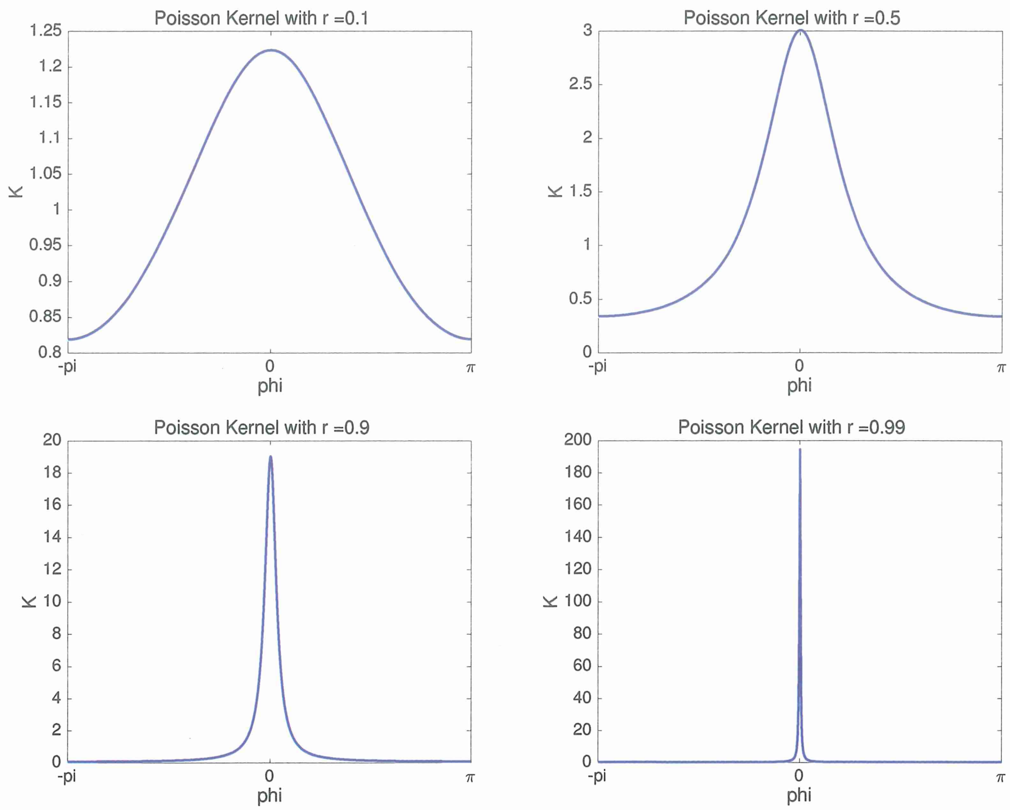
$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos(n\theta) = \operatorname{Re}\left[\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right]$$

$$= \operatorname{Re}\left[\frac{1}{2} + \frac{z}{1-z}\right] = \operatorname{Re}\left[\frac{1+z}{2(1-z)}\right]$$

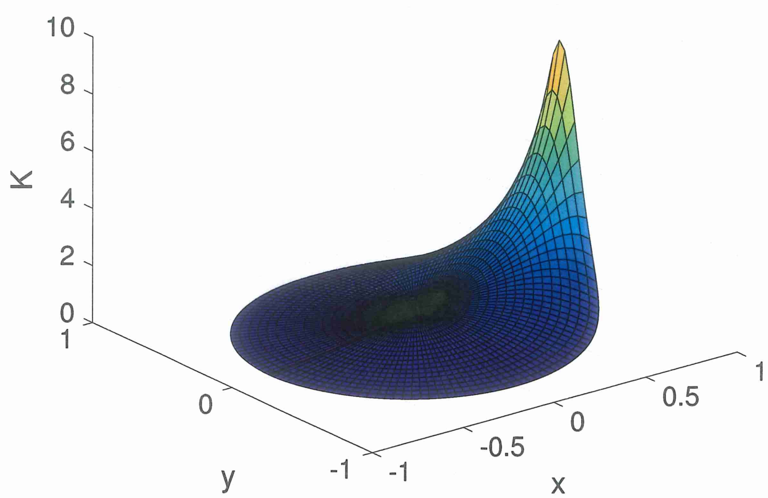
$$= \operatorname{Re}\left[\frac{(1+z)(1-\bar{z})}{2|1-z|^2}\right] = \operatorname{Re}\left[\frac{1+(z-\bar{z})-|z|^2}{2|1-z|^2}\right]$$

$$= \frac{1-r^2}{2(1+r^2-2r\cos\theta)} \qquad \square$$

Poisson Kernel



Poisson Kernel on r <0.8

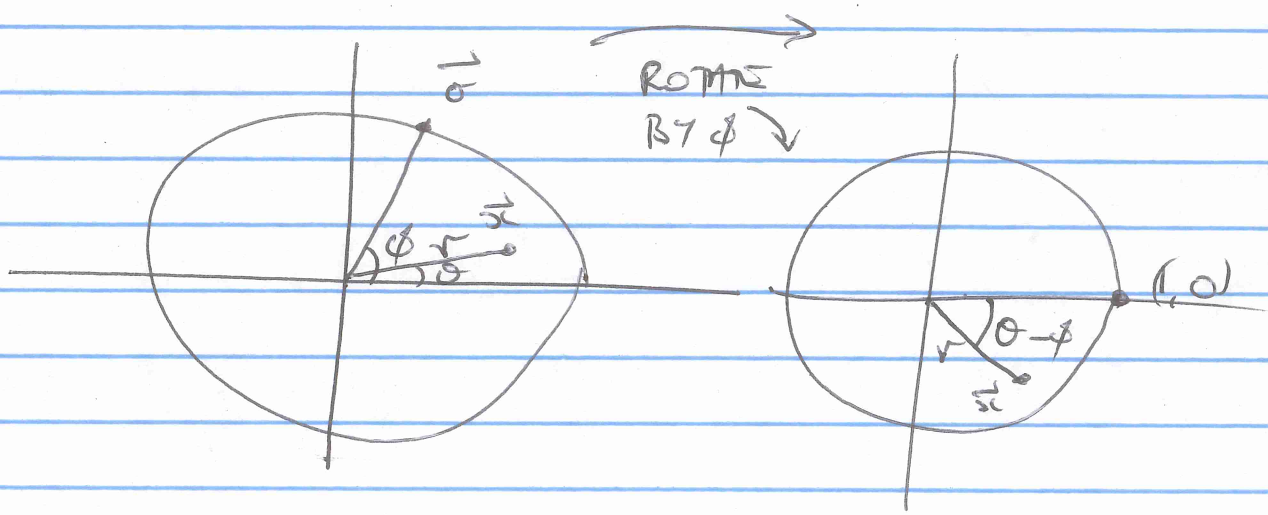


Since ~~where~~

$$K(r,\phi) = \frac{1-r^2}{1+r^2-2r\cos\phi} \qquad \text{POISSON KERNEL}$$

LET $\vec{\sigma} = (\cos\phi, \sin\phi) \in \partial D$

$\vec{x} = (r\cos\theta, r\sin\theta) \in D$



Then

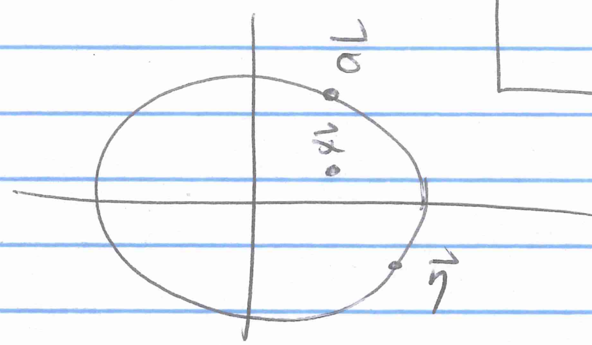
$$\begin{aligned}
|\vec{\sigma}-\vec{x}|^2 &= |(\cos\phi, \sin\phi) - (r\cos\theta, r\sin\theta)|^2 \\
&= |(1,0) - (r\cos(\theta-\phi), r\sin(\theta-\phi))|^2 \\
&= |(1-r\cos(\theta-\phi)), -r\sin(\theta-\phi))|^2 \\
&= 1-2r\cos(\theta-\phi)+r^2
\end{aligned}$$

So denominator is $(\text{distance})^2$ from $\vec{\sigma}$ to $\vec{x}$.

So instead of $K(r, \theta - \phi)$ (polar coords) get

$$K(\vec{x}, \vec{\sigma}) = \frac{1 - |\vec{x}|^2}{|\vec{\sigma} - \vec{x}|^2} \qquad \text{RECT COORDS}$$

$$\boxed{u(\vec{x}) = \frac{1}{2\pi} \int_{S1} h(\vec{y})\, K(\vec{x}, \vec{y})\, d\sigma(\vec{y})}$$



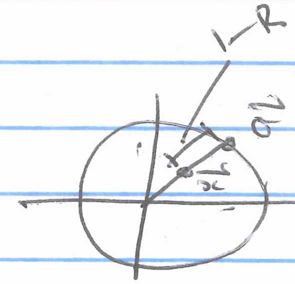
① IF $\vec{x} \to \vec{\eta}$ where $\vec{\eta} \in \partial D$ with $\vec{\eta} \neq \vec{\sigma}$

then $|\vec{\sigma} - \vec{x}| \to |\vec{\sigma} - \vec{\eta}| \neq 0$

and $1 - |\vec{x}|^2 \to 1 - \eta^2 = 0$

So $K(\vec{\sigma}, \vec{x}) \to 0.$

② If $\vec{x} \to \vec{\sigma}$ then $K(\vec{\sigma}, \vec{x}) \to +\infty$.



EX $\vec{x} = R\vec{\sigma}$ with $R \to 1$ gives

$$K(R\vec{\sigma}, \vec{\sigma}) = \frac{1 - R^2}{(1-R)^2} = \frac{1+R}{1-R} \longrightarrow \infty \quad \text{as } R \to 1$$

So it looks like

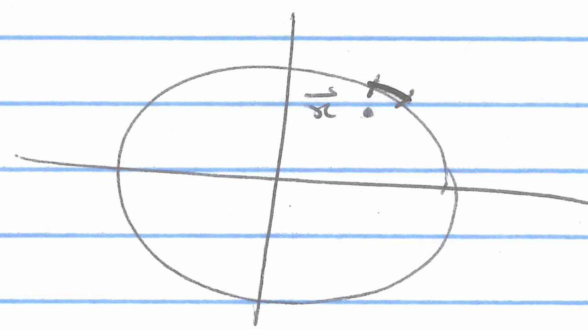
As $r \to 1$, $\quad K_r(\phi) = K(r,\phi) \longrightarrow \delta_0(\phi)$

$\delta_0 =$ δ-FUNCTION AT $\phi = 0$ on S^1

So for r near 1

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\, K(r, \theta-\phi)\, d\phi$$

depends most on values of Boundary Data h at points on circle nearest (r,θ)



But at $r = 0$

$K(0,\phi) = 1$

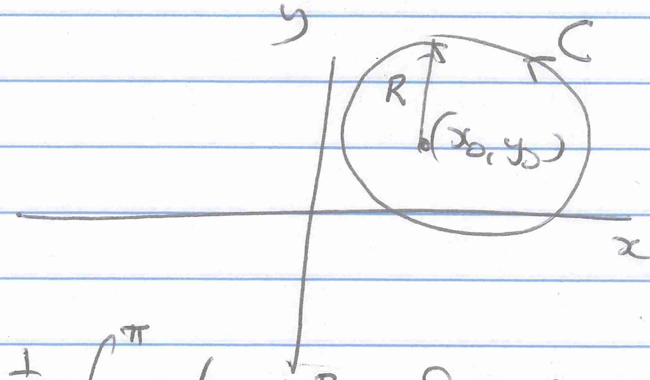
So

$$\underset{\substack{\uparrow \\ \text{ORIGIN}}}{u(\vec{0})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\, d\phi = \text{Average of } h \text{ on circle}$$

This is not an accident:

MEAN VALUE THM

Suppose $\Delta u = 0$



Then

$$u(x_0, y_0) = \frac{1}{2\pi R} \int_{C_R} u \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta$$

Value of u at center of circle equals average value of u around circle.

STRONG MAXIMUM PRINCIPLE

Let Ω be a bounded connected open set in $\mathbb{R}^2$. Suppose $\Delta u = 0$ in Ω, ~~and~~ u is continuous on $\Omega \cup \partial\Omega$ and u is NOT constant.

Let
$$m = \min \{ u(x,y) \,/\, (x,y) \in \partial\Omega \}$$
$$M = \max \{ u(x,y) \,/\, (x,y) \in \partial\Omega \}$$

Then for all (x,y) interior to Ω we have

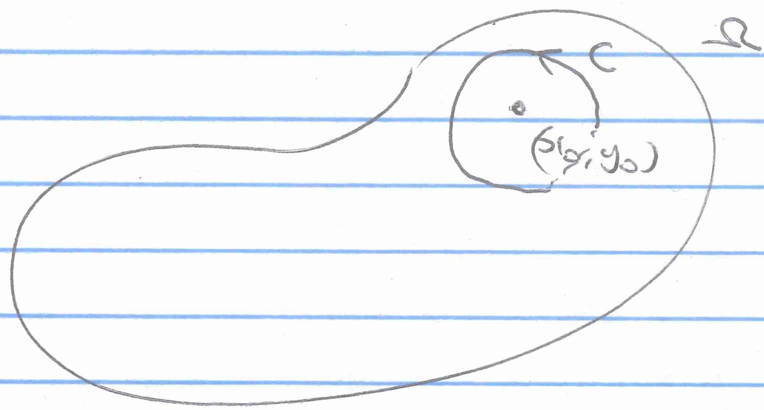
$$m < u(x,y) < M.$$

ie Max + Min of u are achieved ONLY on $\partial\Omega$.

NOTE $\Omega \cup \partial\Omega$ is a CLOSED, BOUNDED set, $u : \Omega \cup \partial\Omega \to \mathbb{R}$ is CB (21)
So u has global MAX + MIN on $\Omega \cup \partial\Omega$.

PROOF IDEA

Suppose u achieves its max at pt (x_0, y_0) interior to Ω



Choose small circle, C, center (x_0, y_0) in Ω.

$u(x_0, y_0) > u(x, y) \quad \forall (x, y) \in C.$
as u NOT CONST.

So $u(x_0, y_0) >$ Average of u on C

$\times$ to Mean Value Thm.

PHYSICAL INTUITION

Drum membrane cannot have internal ~~local~~ max/min if it is at equilibrium.

THM ON UNIQUENESS OF SOLNS

Let Ω be a bounded, connected open set
If

$$\Delta u_1 = f$$
$$\Delta u_2 = f \qquad \text{in } \Omega$$

and $\quad u_1 = u_2 \quad$ on $\partial\Omega$

Then $\quad u_1 = u_2 \quad$ in Ω.

PF

Let $\quad v = u_1 - u_2$.

$\Delta v = 0 \quad$ in Ω

$v = 0 \quad$ on $\partial\Omega$.

By Max Principle min and max of v occur on $\partial\Omega$
Since $v = 0$ on $\partial\Omega$, min and max of v are zero
So $v \equiv 0$. $\square$