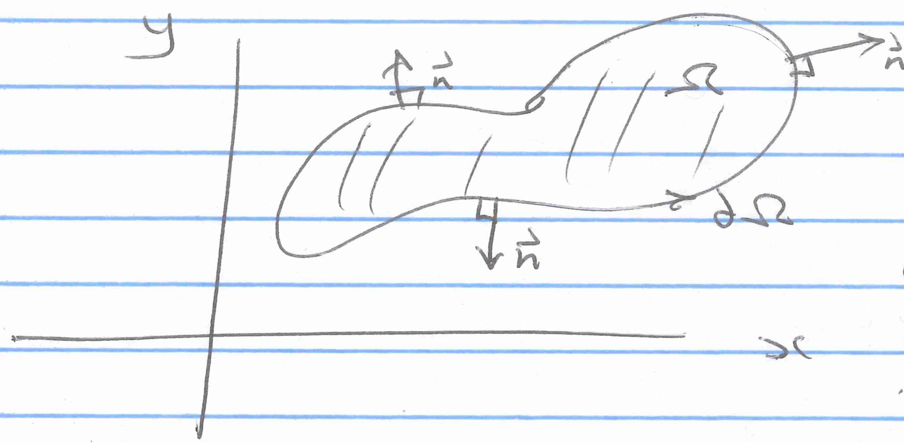
LECTURE 14 SEPARATION OF VARIABLES : POISSON'S EQN

Work in $\mathbb{R}^2$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$u = u(x,y)$$

Work on a domain $\Omega \subseteq \mathbb{R}^2$ with boundary $\partial\Omega$



$\vec{n}$ = UNIT OUTWARD NORMAL ON $\partial\Omega$

BVP FOR POISSON EQN Given $f = f(x,y)$ on Ω

Find $u = u(x,y)$ so that

$$-\Delta u = f \quad \text{on } \Omega$$

- TIME INDEPT
- EQUILIBRIUM PDE

WITH ~~and~~ either

Given on $\partial\Omega$

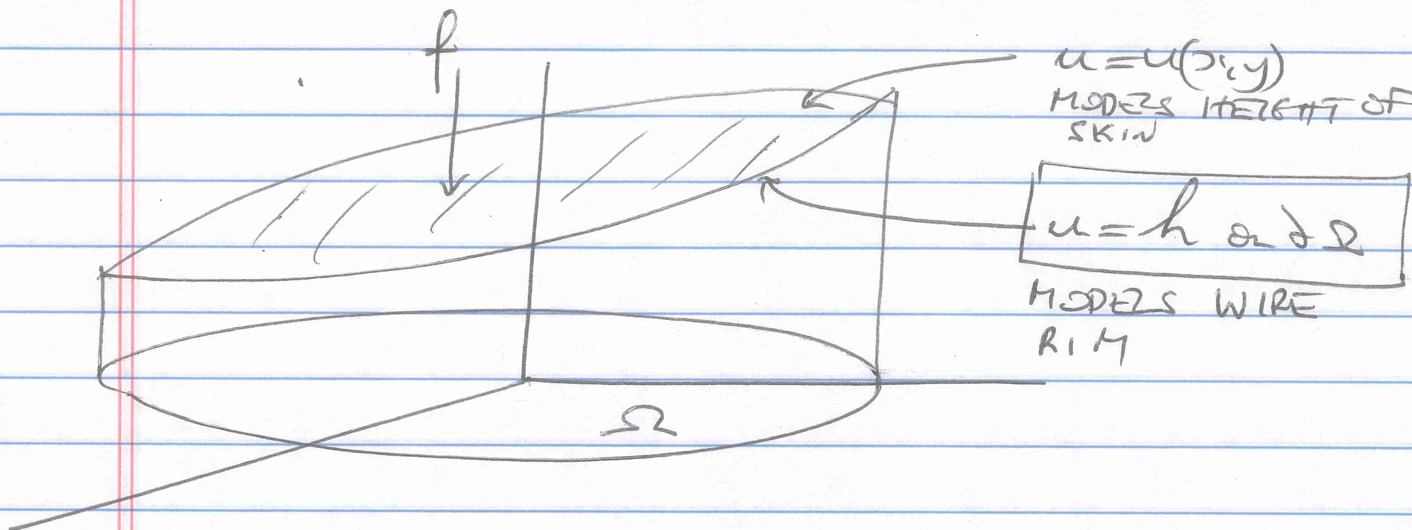
DIRICHLET BC : $u(x,y) = h(x,y)$, $(x,y) \in \partial\Omega$

NEUMANN BC $\dfrac{\partial u}{\partial \vec{n}} \overset{\text{DEF}}{=} \nabla u \cdot \vec{n} = k$ on $\partial\Omega$

$k = k(x,y)$ GIVEN

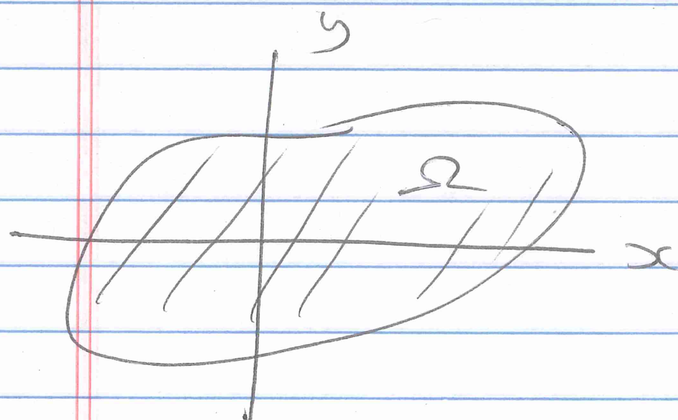
APPLICATIONS

1. EQUILIBRIUM DISPLACEMENT OF DRUM MEMBRANE (SKIN)



$f = f(x,y)$ Models STATIC force applied over Ω.

2.



Ω = HOT PLATE

f = Heat source

u = Temp

DIRICHLET Fix Temp on $\partial\Omega$

NEUMANN $\dfrac{\partial u}{\partial \vec{n}} = 0$ Means $\partial\Omega$ is insulated

Heat cannot escape thru $\partial\Omega$

SEPARATION OF VARIABLES

ONLY WORKS for simple domains
- Rectangles
- Discs

For arbitrary Ω use either
- Green's functions (#6)
- Numerical Methods (#5)

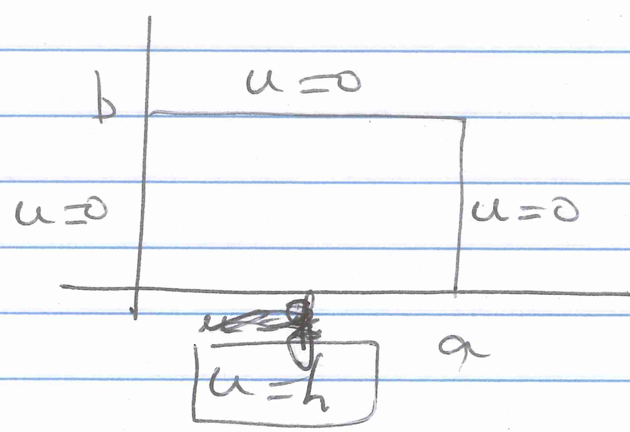
RECTANGULAR DOMAIN

$$\Omega = R = [0, a] \times [0, b]$$

SIMPLE CASE

$$\Delta u = 0 \quad \text{on } R$$

WITH BCs



$$\boxed{\begin{aligned} u(x,0) &= h(x) \\ u(x,b) &= 0 \\ u(0,y) &= 0 \\ u(a,y) &= 0 \end{aligned}}$$

Search for Separable Solⁿ

$$u(x,y) = v(x)\, w(y)$$

$\Delta u = 0$ gives

$$\frac{V''(x)}{V(x)} = -\frac{W''(y)}{W(y)} = \lambda$$

2 ODE's

$$\begin{cases} V'' - \lambda V = 0 \quad \text{on} \quad [0, a] \\ V(0) = V(a) = 0 \end{cases} \qquad 1$$

$$\begin{cases} W'' + \lambda W = 0 \quad \text{on} \quad [0, b] \\ W(b) = 0 \end{cases} \qquad 2$$

NOTE

$$0 = u(x, b) = V(x)\, W(b) \implies W(b) = 0$$

BUT

$$h(x) = u(x, 0) = V(x)\, W(0)$$

cannot hold as V already satisfies 1

To satisfy the non-zero BC we will need to use Fourier Series sol^n built from our separable solutions

We already solved 1 in Lecture 13.
to get

$$\lambda_n = -\frac{n^2\pi^2}{L^2} = -\omega_n^2 < 0$$

$$V_n(x) = \sin\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, 3, \ldots$$

Then we have

$$W_n'' + \lambda_n W_n = 0$$

$$W_n'' - \omega_n^2 W_n = 0 \qquad \boxed{\omega_n = \frac{n\pi}{a}}$$

has sol^ns

$$W_n(y) = A e^{\omega_n y} + B e^{-\omega_n y} \qquad (3)$$

The BC $W_n(b) = 0$ gives one eqn between A, B:

$$0 = A e^{\omega_n b} + B e^{-\omega_n b}$$

So

$$B e^{-\omega_n b} = -A e^{\omega_n b}$$

$$\boxed{B = -A e^{+2\omega_n b}}$$

Plug into 3

$$W_n(y) = A\left[e^{\omega_n y} - e^{2\omega_n b} e^{-\omega_n y}\right]$$

This looks like

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

In fact

$$W_n(y) = 2A e^{\omega_n b}\left[\frac{e^{\omega_n (y-b)} - e^{-\omega_n (y-b)}}{2}\right]$$

$$W_n(y) = \underbrace{2A e^{\omega_n b}}_{\text{JUST A CONST}} \sinh\left[\omega_n (y-b)\right]$$

So set

$$\boxed{W_n(y) = \sinh\left[\frac{n\pi (y-b)}{a}\right]}$$

Gives

$$u_n(x,y) = V_n(x)\, W_n(y) = \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi (y-b)}{a}\right)$$

So get

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left[\frac{n\pi(b-y)}{a}\right] \sin\left[\frac{n\pi x}{a}\right]$$

using sinh is odd.

Find c_n using 4th BC

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} \underbrace{\left[c_n \sinh\left(\frac{n\pi b}{a}\right)\right]}_{b_n} \sin\left(\frac{n\pi x}{a}\right)$$

Fourier Sine Series

So let

$$b_n = \frac{2}{a}\int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Then

$$c_n = \frac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)}$$

and

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right)}$$

OR

$$u(x,y) = \sum_{n=1}^{\infty} \left[\frac{b_n \sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \right] \sin\left(\frac{n\pi x}{a}\right)$$

For each $y \in (0,b)$ we have a F.S.S in x

Suppose h is piecewise CTS or more generally that

$$\int_0^a |h(x)|\,dx < \infty$$

Then $\cup$ ✓ $\exists M : \forall n$

$$|b_n| \leq M.$$

So

$$\left| b_n \frac{\sinh\left(\frac{n\pi(b-y)}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \right| \leq M \frac{\sinh \frac{n\pi(b-y)}{a}}{\sinh\left(\frac{n\pi b}{a}\right)}$$

$$= M \frac{e^{\frac{n\pi(b-y)}{a}} - e^{-\frac{n\pi(b-y)}{a}}}{e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}}$$

$$= M \frac{e^{-\frac{n\pi(b-y)}{a}}\left[e^{\frac{2n\pi(b-y)}{a}} - 1\right]}{e^{-\frac{n\pi b}{a}}\left[e^{\frac{2n\pi b}{a}} - 1\right]}$$

n LARGE

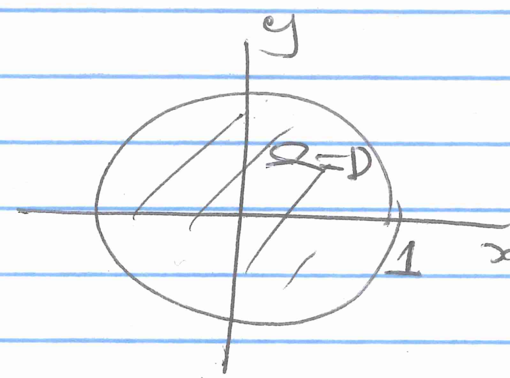
$$\cong M e^{\frac{n\pi y}{a}} e^{-\frac{2n\pi y}{a}} = M e^{-\frac{n\pi y}{a}} \longrightarrow 0$$

as $n \to \infty$ $\forall y$.

IN FACT can differentiate term by term

$\boxed{\text{SEPN OF VARIABLES IN POLAR COORDS}}$

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$



~~GIVEN $h = h(\theta)$ have~~

$$\cancel{h(x,y) = h(\cos\theta, \sin\theta)}$$

WORK in POLAR COORDS

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Write $u_{PC} = u_{PC}(r,\theta)$, $u_{RC} = u_{RC}(x,y)$

Then
$$\boxed{u_{PC}(r,\theta) = u_{RC}(r\cos\theta, r\sin\theta)}$$

By Chain Rule

$$\frac{\partial u_{PC}}{\partial r} = \frac{\partial u_{RC}}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u_{RC}}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial u_{PC}}{\partial r} = \cos\theta\frac{\partial u_{RC}}{\partial x} + \sin\theta\frac{\partial u_{RC}}{\partial y}$$

OR MORE SIMPLY

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

and
$$\frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

MATRIX–VECTOR FORM

$$\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix}$$

Inverting Matrix eqn gives

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}}$$

SECOND DERIVATIVES

$$\frac{\partial^2 u_{RC}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u_{pc}}{\partial x}\right)$$

$$= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]\left[\cos\theta \frac{\partial u_{pc}}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u_{pc}}{\partial \theta}\right]$$

$$= \cos^2\theta \frac{\partial^2 u_{pc}}{\partial r^2} - \cos\theta \sin\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_{pc}}{\partial \theta}\right)$$

$$\cancel{- \cos\theta \sin\theta \frac{1}{r}\frac{\partial}{\partial \theta}}$$

$$- \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial u_{pc}}{\partial r}\right] + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u_{pc}}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 u_{pc}}{\partial r^2} + \cos\theta \sin\theta \frac{1}{r^2}\frac{\partial u_{pc}}{\partial \theta} - \cos\theta \sin\theta \frac{1}{r}\frac{\partial^2 u_{pc}}{\partial \theta^2}$$

INVERTING gives

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}}$$

SECOND DERIVATIVES

CAUTION: PUT u BACK IN!!

$$\frac{\partial^2 u_{RC}}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u_{RC}}{\partial x}\right]$$

$$= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]\left[\cos\theta \frac{\partial u_{PC}}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u_{PC}}{\partial \theta}\right]$$

$$= \cos^2\theta \frac{\partial^2 u_{PC}}{\partial r^2} - \cos\theta\sin\theta \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial u_{PC}}{\partial \theta}\right]$$

$$- \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial u_{PC}}{\partial r}\right] + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial u_{PC}}{\partial \theta}\right]$$

$$= \cos^2\theta \frac{\partial^2 u_{PC}}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u_{PC}}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u_{PC}}{\partial r\partial \theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u_{PC}}{\partial r} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u_{PC}}{\partial \theta\partial r}$$