LECTURE 13 SEPARATION OF VARIABLES: THE WAVE EQN

WAVE EQUATION FOR $u = u(t, x)$:

$$u_{tt} = c^2 u_{xx}. \quad (1)$$

SOLUTION METHOD depends on whether domain is all of $\mathbb{R}$ or a finite interval:

$x \in \mathbb{R}$: Use d'Alembert's Solution

$x \in [a, b]$: Use Separation of Variables + Fourier Series.

SEPARATION OF VARIABLES

Search for solutions of form

$$u(t, x) = W(t) V(x)$$

Since $u_{tt} = W'' V$ and $u_{xx} = W V''$
we get

$$W'' V = c^2 W V''$$

$$\frac{W''(t)}{W(t)} = c^2 \frac{V''(x)}{V(x)} = \lambda$$

$\lambda =$ CONSTANT INDEPT of t (BY RHS) and x (by LHS)

UPSHOT $w = w(t)$ and $v = v(x)$ must both satisfy ODEs:

$$w''(t) - \lambda w(t) = 0$$

$$v''(x) - \frac{\lambda}{c^2} v(x) = 0.$$

$\therefore$ GET FOLLOWING PAIRS OF LINEARLY INDEPENDENT SEPARABLE SOLUTIONS

λ	$w(t)$	$v(x)$	$u(t,x)$
$\lambda = -\omega^2 < 0$	$\cos(\omega t)$ $\sin(\omega t)$	$\cos\left(\frac{\omega x}{c}\right)$ $\sin\left(\frac{\omega x}{c}\right)$	$\cos(\omega t)\cos\left(\frac{\omega x}{c}\right)$ $\cos(\omega t)\sin\left(\frac{\omega x}{c}\right)$ $\sin(\omega t)\cos\left(\frac{\omega x}{c}\right)$ $\sin(\omega t)\sin\left(\frac{\omega x}{c}\right)$
$\lambda = 0$	$1,\ t$	$1,\ x$	$1,\ t,\ x,\ tx$
$\lambda = \omega^2 > 0$	$e^{-\omega t}$ $e^{+\omega t}$	$e^{-\frac{\omega x}{c}}$ $e^{\frac{\omega x}{c}}$	$e^{-\omega(t + x/c)}$ $e^{-\omega(t - x/c)}$ $e^{\omega(t + x/c)}$ $e^{\omega(t - x/c)}$

ALSO $\lambda = (\pm\gamma)^2 \in \mathbb{C}$ gives

$w(t) = e^{\gamma t}, e^{-\gamma t}$

$v(x) = e^{\gamma x/c}, e^{-\gamma x/c}$

③

TERMINOLOGY ABOUT WAVES (CASE $\lambda = -\omega^2 < 0$)

① ω = Angular Frequency. UNITS $= \frac{\text{RAD}}{\text{TIME}}$

as ωt in $\cos(\omega t)$ must have units of radians

② $f = \frac{\omega}{2\pi}$ = Frequency

= # Periods in 1 unit of time at fixed x.

UNITS : $\frac{1}{\text{TIME}}$ NOTE s^{-1} = Hz = HERTZ

③ c = Wave Speed UNITS $= \frac{\text{LENGTH}}{\text{TIME}}$

④ $k := \frac{\omega}{c}$ = Wave Number

= # Periods in Spatial Interval of Length 2π

REASON $\cos\left(\frac{\omega x}{c}\right) = \cos(kx)$ has k periods in 2π.

⑤ $\lambda = \frac{2\pi}{k}$ = Wavelength = Period in x-variable

SOME SIMPLE (NON-SEPARABLE) SOLUTIONS

① $\lambda = -\omega^2 < 0$.

$$\cos(kx - \omega t) = \cos\left(\frac{\omega x}{c} - \omega t\right)$$

$$= \cos\left(\frac{\omega x}{c}\right)\cos(\omega t) + \sin\left(\frac{\omega x}{c}\right)\sin(\omega t)$$

is also a solution of wave eqn (by linearity)

BUT This sol^n is NOT separable

② Same for $\sin(kx - \omega t)$

③ $e^{(kx - \omega t)} = e^{-\omega\left(t + \frac{x}{c}\right)}$ is also a solution.

It is separable though!

$\boxed{\text{CHOOSING } \lambda \text{ TO SATISFY BCs}}$

$\boxed{\text{ZERO DIRICHLET BCs}}$

~~Use~~ FOR $x \in [0, L]$ suppose have

$$u(t, 0) = 0 = u(t, L)$$

VIBRATING STRING FIXED AT BOTH ENDS

Since $u(t,x) = w(t)\,v(x)$ we get

$$w(t)\,v(0) = 0$$
$$w(t)\,v(L) = 0$$

To ensure we get a non-zero sol^n we need $w(t) \neq 0$. So get

$$\boxed{v(0) = 0 = v(L)}$$

NOW G.S. of

$$v'' - \frac{\lambda}{c^2} v = 0$$

is of form

$$v(x) = A e^{\frac{\gamma x}{c}} + B e^{-\frac{\gamma x}{c}} \quad \text{where } \lambda = \gamma^2, \ \gamma \in \mathbb{C}.$$

Then

$$0 = v(0) = A + B \quad \Rightarrow \quad B = -A$$
$$0 = v(L) = A e^{\gamma L/c} + B e^{-\gamma L/c}$$

So $A\left(e^{\gamma L/c} - e^{-\gamma L/c}\right) = 0$

Since we want $v \not\equiv 0$, we need $A, B \neq 0$.

This forces $e^{\gamma L/c} = e^{-\gamma L/c}$

or $$e^{2\gamma L/c} = 1$$

Write $$\gamma = \rho + i\omega$$

Then $$e^{\frac{2L}{c}\rho}\, e^{i\frac{2L}{c}\omega} = 1 \qquad (*)$$

Since $\left|e^{i\frac{2L}{c}\omega}\right| = 1$ we get

$$e^{\frac{2L}{c}\rho} = 1 \Rightarrow \rho = 0$$

So $\gamma = i\omega$ is pure imaginary

and $\lambda = -\omega^2 \leq 0$.

NEXT since $\rho = 0$ by $(*)$ we have

$$\cos\left(\frac{2L}{c}\omega\right) = 1$$

so $$\frac{2L}{c}\omega = 2n\pi$$

or $$\omega_n = \frac{n\pi c}{L}$$

$$\boxed{\lambda_n = -\left(\frac{n\pi c}{L}\right)^2} \quad \text{and} \quad \boxed{V_n(x) = \sin\left(\frac{n\pi x}{L}\right)}$$

From our pair of separable solutions we get

$$W_n(t) = \cos\left(\frac{n\pi c t}{L}\right)$$

$$\tilde{W}_n(t) = \sin\left(\frac{n\pi c t}{L}\right)$$

2ND FACTORS SAME

and

$$u_n(t,x) = W_n(t)\, v_n(x) = \cos\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$\hat{u}_n(t,x) = \hat{W}_n(t)\, v_n(x) = \sin\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Taking linear combinations get G.S.

$$u(t,x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + d_n \sin\left(\frac{n\pi c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty}\left[b_n \cos\left(\frac{n\pi c t}{L}\right) + d_n \sin\left(\frac{n\pi c t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \qquad (*)$$

— A FOURIER SINE SERIES IN x

with time-dependent coefficients.

$\boxed{\text{To find } b_n, d_n \text{ use IC's}}$

SUPPOSE
$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x)$$

PLUG into ⊛ to get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{F.S.S. for } f,$$

So
$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)}$$

AND
$$g(x) = u_t(0,x)$$
$$g(x) = \sum_{n=1}^{\infty} d_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

So
$$d_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\boxed{d_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

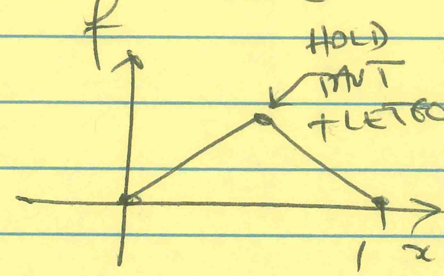
EX

$$\begin{cases} u_{tt} = u_{xx} \quad \text{ON} \quad [0,1] \\ u(t,0) = 0 = u(t,1) \\ u(0,x) = f(x) \\ u_t(0,x) = 0 \end{cases}$$

where

$$f(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1-x & \frac{1}{2} \le x \le 1 \end{cases}$$

IC



Previously we found

$$b_{2k+1} = \frac{4(-1)^k}{(2k+1)^2 \pi^2}$$

$$b_{2k} = 0$$

ALSO d_n are got from F.S. of $g(x) \equiv 0$

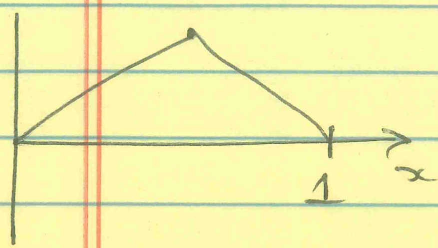
so $d_n = 0 \ \forall n$

SOLN

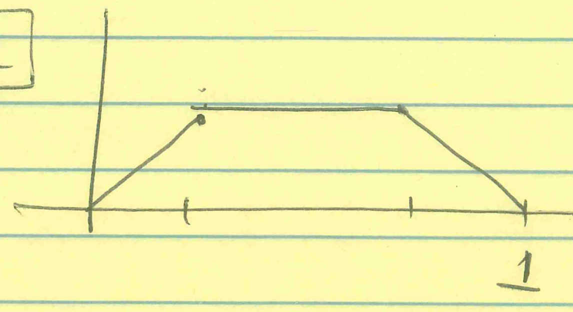
$$u(t,x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \frac{\cos[(2k+1)\pi t] \sin[(2k+1)\pi x]}{(2k+1)^2}$$

GRAPH OF SOLN

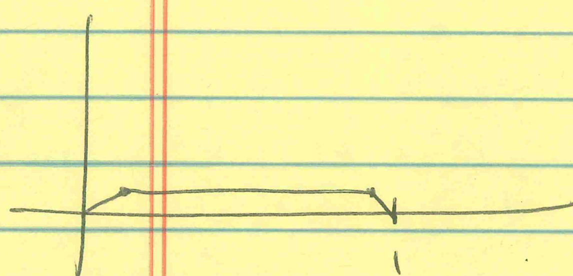
$\boxed{t=0}$



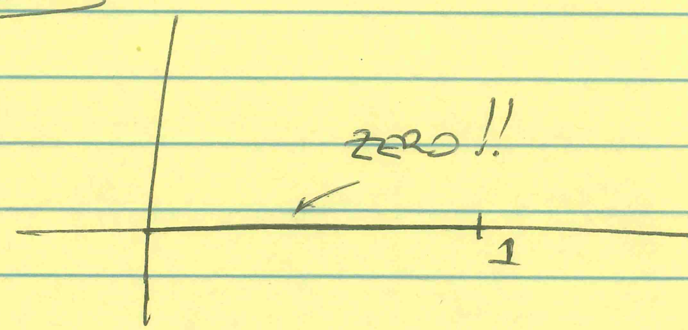
$\boxed{t=0.2}$



$\boxed{t=0.4}$



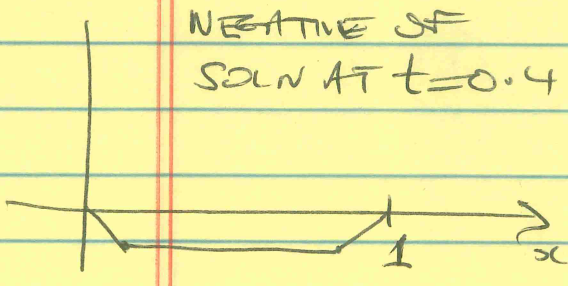
$\boxed{t=0.5}$



As $\cos\left((2k+1)\frac{\pi}{2}\right) = 0 \quad \forall k$

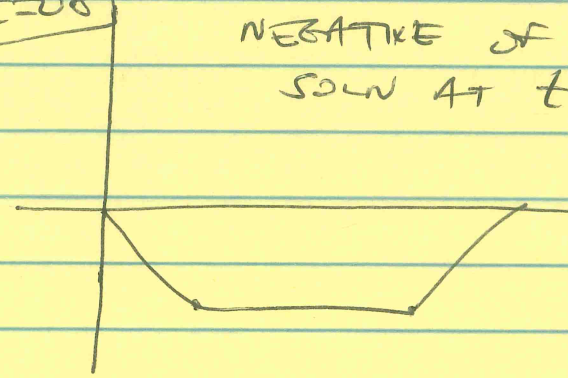
$\boxed{t=0.6}$

NEGATIVE OF SOLN AT $t=0.4$



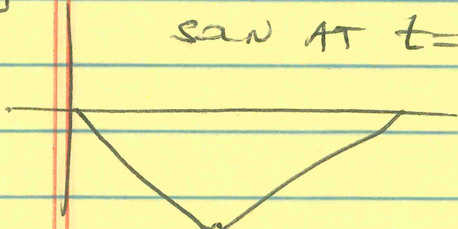
$\boxed{t=0.8}$

NEGATIVE OF SOLN AT $t=0.2$



$\boxed{t=1}$

NEGATIVE OF SOLN AT $t=0$.



$\boxed{\text{AT } t=2}$ SOLUTION IS SAME AS AT $t=0$.

NOTE Wave eqn does NOT smooth out non-smooth initial data, unlike heat eqn.

From graphs above, we ~~notice~~ GUESS following symmetry

CLAIM

Whenever we have ZERO DIRICHLET BCs

1. $u(t+\frac{L}{c}, x) = -u(t, L-x)$
2. So by applying 1 twice we get

$$u(t+\frac{2L}{c}, x) = u(t, x)$$

has period $\frac{2L}{c}$ in time.

$\boxed{\text{PF OF 1}}$

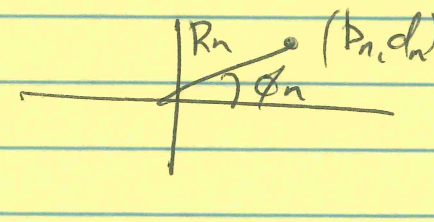
INITIAL WAVE REPRODUCED 1ST AS UPSIDE DOWN MIRROR IMAGE OF ITSELF AT $t = L/c$, THEN IN ORIGINAL FORM AT $t = 2L/c$.

We have

$$u(t,x) = \sum_{n=1}^{\infty} \left[b_n \cos\left(\frac{n\pi c t}{L}\right) + d_n \sin\left(\frac{n\pi c t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

Now if we set

$$R_n = \sqrt{b_n^2 + d_n^2}$$
$$\phi_n = \arctan\left(d_n / b_n\right)$$



Then

$$R_n \cos\left(\frac{n\pi ct}{L} \mp \phi_n\right)$$

$$= R_n \cos\phi_n \cos\left(\frac{n\pi ct}{L}\right) + R_n \sin\phi_n \sin\left(\frac{n\pi ct}{L}\right)$$

$$= b_n \cos\left(\frac{n\pi ct}{L}\right) + d_n \sin\left(\frac{n\pi ct}{L}\right)$$

So

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi ct}{L} - \phi_n\right) \sin\left(\frac{n\pi x}{L}\right)}$$

So

$$u\left(t+\frac{L}{c}, x\right) = \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi ct}{L} - \phi_n + n\pi\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} R_n \left[\cos\left(\frac{n\pi ct}{L} - \phi_n\right) \cos(n\pi) - \sin\left(\frac{n\pi ct}{L} - \phi_n\right) \cancel{\sin(n\pi)}\right] \times \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi ct}{L} - \phi_n\right) \left[\cos n\pi \sin\left(\frac{n\pi x}{L}\right) - \sin n\pi \cos\left(\frac{n\pi x}{L}\right)\right]$$

$$= \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi ct}{L} - \phi_n\right) \sin\left(\frac{n\pi x}{L} - n\pi\right)$$

$$= -\sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi c t}{L} - \phi_n\right) \sin\left(\frac{n\pi}{L}(L-x)\right)$$

$$= -u(t, L-x)$$ ✓

D.