LECTURE 12 SEPARATION OF VARIABLES: HEAT EQN II

$\boxed{\text{HEATED RING}}$

$$\begin{cases} u_t = u_{xx} & -\pi \le x \le \pi, \quad t > 0 \\ u(t,-\pi) = u(t,\pi) & \\ u_x(t,-\pi) = u_x(t,\pi) & \\ u(0,x) = f(x) & \end{cases}$$

$\boxed{\text{STEP I}}$

SUPPOSE $u(t,x) = e^{\lambda t} v(x)$

Then $u_t = u_{xx}$ gives $v'' + \lambda v = 0$

BCs give $v(-\pi) = v(\pi)$, $v'(-\pi) = v'(\pi)$

$\boxed{\text{CASE } \lambda \geq 0}$ NO SOLNS (USUAL)

$\boxed{\text{CASE } \lambda = -\omega^2 < 0}$

$$v(x) = A\cos(\omega x) + B\sin(\omega x)$$

$$v(-\pi) = v(\pi) \implies A\cos(\omega\pi) - B\sin(\omega\pi) = A\cos(\omega\pi) + B\sin(\omega\pi)$$

$$\implies 2B\sin(\omega\pi) = 0$$

$$\implies B = 0 \quad \text{OR} \quad \omega = n \in \mathbb{N}.$$

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$\boxed{B=0}$ $V = A\cos(\omega x)$

$V' = -A\omega \sin(\omega x)$

$V'(-\pi) = V'(\pi) \overset{\text{as BY}}{\Longrightarrow} \cancel{A=0}$ or $\omega = n$

So get $\boxed{V(x) = \cos(nx)}$

OR

$\boxed{\omega = n}$

$V(x) = A\cos(nx) + B\sin(nx)$

ALREADY HAVE THIS

So have

$\boxed{\tilde{V}(x) = \sin(nx)}$

$\boxed{\text{STEP II}}$

So eigensolutions are

$u_n(t,x) = e^{-n^2 t}\cos(nx) \qquad n = 0,1,2,\ldots$

$\tilde{u}_n(t,x) = e^{-n^2 t}\sin(nx) \qquad n = \cancel{0},1,2,\ldots$

General Sol^n is FS

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos(nx) + b_n e^{-n^2 t}\sin(nx)$$

where a_n, b_n are given by Initial Data:

$$f(x) = u(0, x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

So

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

For large enough $t > 0$

$$u(t, x) \simeq \frac{a_0}{2} + a_1 e^{-t} \cos x + b_1 e^{-t} \sin x$$

$$\xrightarrow{t \to \infty} \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \text{AVERAGE INITIAL TEMP.}$$

PHYSICALLY Since RING Structure prevents heat from escaping (no ends!) temp converges rapidly to a constant.

INHOMOGENEOUS BCs

SOLVE

$$\begin{cases} u_t = u_{xx} & 0 \leq x \leq 1, \quad t > 0 \\ u(t, 0) = \alpha & \\ u(t, 1) = \beta . & \end{cases}$$

α, β CONSTANTS (indep't of t)

IDEA: CONVERT This Problem into a problem with HOMOGENEOUS BCs ($\alpha = \beta = 0$)

TRICK: EQUILIBRIUM TEMPERATURE

$$u^*(x) := \lim_{t \to \infty} u(t, x)$$

NOTICE

$$0 = \frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial x^2}$$

So $u^*(x) = mx + b$ is linear

ALSO $u^*(0) = \alpha$, $u^*(1) = \beta$ must hold

So get

$$u^*(x) = (\beta - \alpha)x + \alpha = \beta x + \alpha(1 - x).$$

NEXT SET $\boxed{v(t,x) = u(t,x) - u^*(x)}$

Then v solves

$$\begin{cases} v_t = v_{xx} \\ v(t,0) = 0 \\ v(t,1) = 0 \\ v(0,x) = f(x) - u^*(x) =: \hat{f}(x) \end{cases}$$

We already know how to solve for v by LECT 11.

Get

$$u(t,x) = u^*(x) + v(t,x)$$

$$\boxed{u(t,x) = \beta x + \alpha(1-x) + \sum_{n=1}^{\infty} \tilde{b}_n \exp[-n^2\pi^2 t] \sin[n\pi x]}$$

where $\boxed{\tilde{b}_n = 2\int_0^1 \hat{f}(x) \sin(n\pi x)\, dx}$