LECTURE 11 SEPARATION OF VARIABLES: HEAT EQUATION

Let $u = u(t,x)$ = Temperature in a 1D bar

HEAT EQN $\qquad \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} \qquad t \geq 0, \quad a \leq x \leq b$

$k > 0$ IS THERMAL DIFFUSIVITY

INITIAL CONDITION $\qquad u(0,x) = f(x) \qquad f$ GIVEN

BOUNDARY CONDITIONS

3 COMMON OPTIONS

(A) DIRICHLET (Prescribe Temp at ends)

$$u(t,a) = \alpha(t), \qquad u(t,b) = \beta(t)$$

(B) NEUMANN (Prescribed Heat Flux across ends)

$$\frac{\partial u}{\partial x}(t,a) = \mu(t) \qquad \frac{\partial u}{\partial x}(t,b) = \nu(t)$$

Zero Neumann Cond^{n} $\Longleftrightarrow$ Insulated ends

(C) PERIODIC (Heated Ring)

$$u(t,a) = u(t,b), \qquad \frac{\partial u}{\partial x}(t,a) = \frac{\partial u}{\partial x}(t,b)$$

FOCUS ON

$$\begin{cases} u_t = k\, u_{xx} & t \geq 0, \quad 0 \leq x \leq L \\ u(0,x) = f(x) \\ u(t,0) = 0 = u(t,L) \end{cases}$$

Search for <u>SEPARABLE SOLUTIONS</u>

$$\boxed{u(t,x) = w(t)\, v(x)}$$

Product of f^n of t with f^n of x

Then $\quad u_t = k u_{xx} \quad$ gives

$$w_t v = k w v_{xx}$$

$$\frac{w_t}{w} = k \frac{v_{xx}}{v}$$

FUNCTION OF t = FUNCTION OF v

Since RHS is indept of t: $\quad w_t = -\lambda w$, for some constant λ.

So $\quad w(t) = C e^{-\lambda t}$

(ABSORB C into v)

$$\therefore \quad \boxed{u(t,x) = e^{-\lambda t}\, v(x)}$$

where $\mathcal{L}[v] = -k v_{xx}$ satisfies

$$\mathcal{L}[v] = \lambda v$$

So we need to find e-pairs of operator $\mathcal{L}$.

$\boxed{\text{BCs GIVE POSSIBLE } \lambda\text{'s}}$

$$0 = u(t,0) = e^{-\lambda t} v(0)$$
$$0 = u(t,L) = e^{-\lambda t} v(L)$$

So have ODE BVP

$$\boxed{-k v_{xx} = \lambda v, \quad v(0) = 0 = v(L)}$$

As in ODEs course let

$$v(x) = e^{\mu x}$$

Plug into ODE: $\quad -k\mu^2 = \lambda$

$\boxed{\text{CASE } \lambda \in \mathbb{R}, \ \lambda \leq 0}$

$$\mu = \pm\sqrt{\frac{|\lambda|}{k}}$$

BC: $\quad v(x) = A e^{\mu x} + B e^{-\mu x} \qquad \mu = \sqrt{\frac{|\lambda|}{k}}$

BC $\boxed{x=0}$ $A+B=0$

$\boxed{x=L}$ $Ae^{\mu L}+Be^{-\mu L}=0$

OR

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since det of matrix is never zero, only sol^n is $A=B=0$

So $\boxed{v(x)=0}$. NOT INTERESTING

CASE $\lambda = a+ib$, $a\neq 0, b\neq 0$

Similarly $\forall$ ✓ only sol^n is $v(x)=0$.

CASE $\lambda \in \mathbb{R}$, $\lambda > 0$

$$\mu^2 = -\frac{\lambda}{k}$$

Let $\mu = i\omega$

Then get $-\omega^2 = -\frac{\lambda}{k}$

OR $\boxed{\omega = \sqrt{\frac{\lambda}{k}}}$ $\omega \in \mathbb{R}$

So

$$v(x) = \tilde{A} e^{i\omega x} + \tilde{B} e^{-i\omega x}$$

or

$$\boxed{v(x) = A\cos\omega x + B\sin\omega x}$$

$\boxed{BC}$ $\boxed{x=0}$ $A = 0.$

So $v(x) = B\sin\omega x$

$\boxed{x=L}$ $B\sin(\omega L) = 0.$

Since $B \neq 0$ (else $v(x) = 0$) we must have

$$\omega L = n\pi \qquad n \in \{1, 2, 3 \ldots\}$$

So $\boxed{\omega = \frac{n\pi}{L}}$ $\lambda = k\omega^2$

SUMMARY

Eigenvalues $\lambda_n = k\left(\frac{n\pi}{L}\right)^2$

Eigenfunctions $v_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

EIGEN SOLUTIONS

$$u_n(t,x) = e^{-\lambda_n t} v_n(x)$$

$$\boxed{u_n(t,x) = e^{-\left(\frac{kn^2\pi^2}{L^2} t\right)} \sin\left(\frac{n\pi x}{L}\right)}$$

REMAINING ISSUE.

We still need to satisfy IC $u(0,x) = f(x)$.

Since the u_n solve heat eqn by superposition so does

$$u(t,x) = \sum_{n=1}^{\infty} b_n u_n(t,x)$$

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} b_n \exp\left[-\frac{kn^2\pi^2}{L} t\right] \sin\left(\frac{n\pi x}{L}\right)}$$

GENERAL SOLN.

Assuming series converges:

$$f(x) = u(0,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

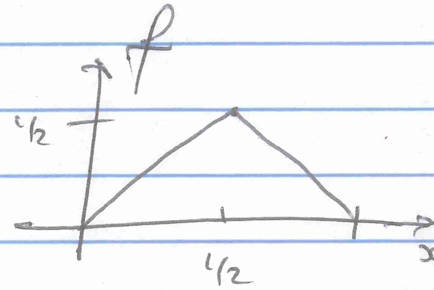
FOURIER SINE SERIES ON $[0,L]$

So

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

EX $\quad u_t = u_{xx} \quad$ [So $k=1$] on $[0,1]$

$$f(x) = \begin{cases} x & \text{IF } 0 \le x < \frac{1}{2} \\ 1-x & \text{IF } 1/2 \le x < 1 \end{cases}$$



$$b_n = 2\int_0^{\frac{1}{2}} x \sin(n\pi x)\,dx + 2\int_{1/2}^{1} (1-x)\sin(n\pi x)\,dx$$

$$b_n = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \qquad b_n = 0 \text{ for } n = 2m \text{ even}$$

Let $n = 2\ell + 1$ be odd.
Then

$$b_{2\ell+1} = \frac{4}{(2\ell+1)^2\pi^2}(-1)^\ell$$

and

$$f(x) = \sum_{\ell=1}^{\infty} \frac{4(-1)^\ell}{(2\ell+1)^2\pi^2}\sin[(2\ell+1)\pi x]$$

Sol^n to Heat Eqn is therefore

$$u(t,x) = \frac{4}{\pi^2}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{(2\ell+1)^2}\sin[(2\ell+1)\pi x]\exp[-(2\ell+1)^2\pi^2 t]$$

$$u(t,x) = \frac{4}{\pi^2} \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l+1)^2} \exp\left[-(2l+1)^2 \pi^2 t\right] \sin\left[(2l+1)\pi x\right]$$

OBSERVATIONS $(k=1,\ L=1)$

① The Sharp corner at $x = \frac{1}{2}$ in initial data f is smoothed out immediately

② This phenomenon occurs whenever the initial data satisfies

$$f \in L^1([0,1]) \text{ in that } \int_0^1 |f(x)|\, dx < \infty.$$

INTUITION @ Then

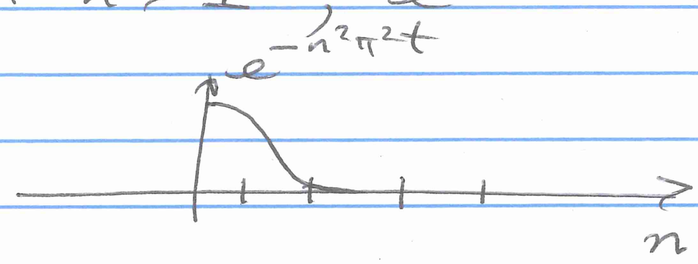
$$|b_n| \le 2\int_0^1 |f(x) \sin(n\pi x)|\, dx$$

$$\le 2\int_0^1 |f(x)|\, dx := M < \infty \quad \forall n \ge 1$$

ⓑ

$\boxed{\text{FIX any } t > 0}$

Then for most $n > 1$, $e^{-n^2\pi^2 t} << 1$ is VERY SMALL



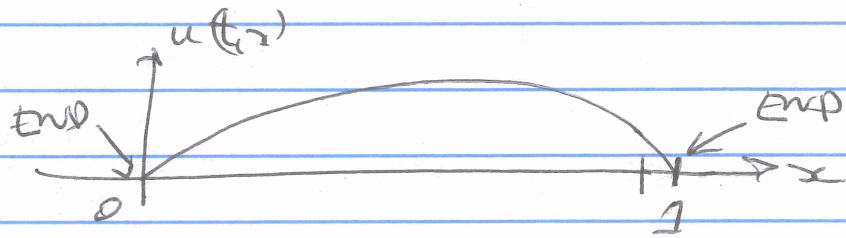
(c) So in the FS

$$u(t,x) = \sum_{n=1}^{\infty} b_n \exp[-n^2\pi^2 t] \sin(n\pi x)$$

for any $t > 0$, only the first few terms matter and rest are essentially zero.

In fact once t is large enough

$$u(t,x) \simeq b_1 \exp[-\pi^2 t] \sin[\pi x]$$



and $\lim_{t\to\infty} u(t,x) = 0$

PHYSICALLY Because we chose to hold temp at ends to be 0, any initial heat energy eventually dissipates (spreads out) thru the ends.

THM

(3) If the initial data $f(x) = u(0,x)$ is L^1

Then for any $t > 0$, $u(t,x)$ is an infinitely differentiable f^n of x

IN PARTICULAR We can choose initial data to be very unsmooth (NOISY). The heat eqn then smooths out noise INSTANTANEOUSLY

- ∃ Applications in Image Processing to use "heat flow" to blur out noise.

ONE WAY to create a noisy f:

Given a smooth $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

add noise by

$$f^{NOISY}(x) = \sum_{n=1}^{\infty} (b_n + z_n) \sin(n\pi x)$$

where

$z_n \sim N(0, \sigma^2)$ are IID Gaussian Random Variables with mean=0 and variance=σ^2.

$\boxed{\text{REASON } u(t,x) \text{ IS SMOOTH for any } t>0}$

$$u(t,x) = \sum_{n=1}^{\infty} b_n \exp[-n^2\pi^2 t] \sin[n\pi x] = \sum_{n=1}^{\infty} v_n(t,x)$$

(A) Apply Weierstrass M-Test to show u is CTS in x $\forall t>0$

Fix $t>0$

We have

$$|v_n(t,x)| \leq M R^{n^2} \qquad R = e^{-\pi^2 t} < 1$$

as $t>0$.

SINCE $\displaystyle\sum_{n=1}^{\infty} R^{n^2} \leq \sum_{m=1}^{\infty} R^m < \infty$ if $R<1$ (GEO SERIES)

$$u(t,x) = \sum_{n=1}^{\infty} v_n(t,x) \quad \text{CONVS UNIF + ABS}$$

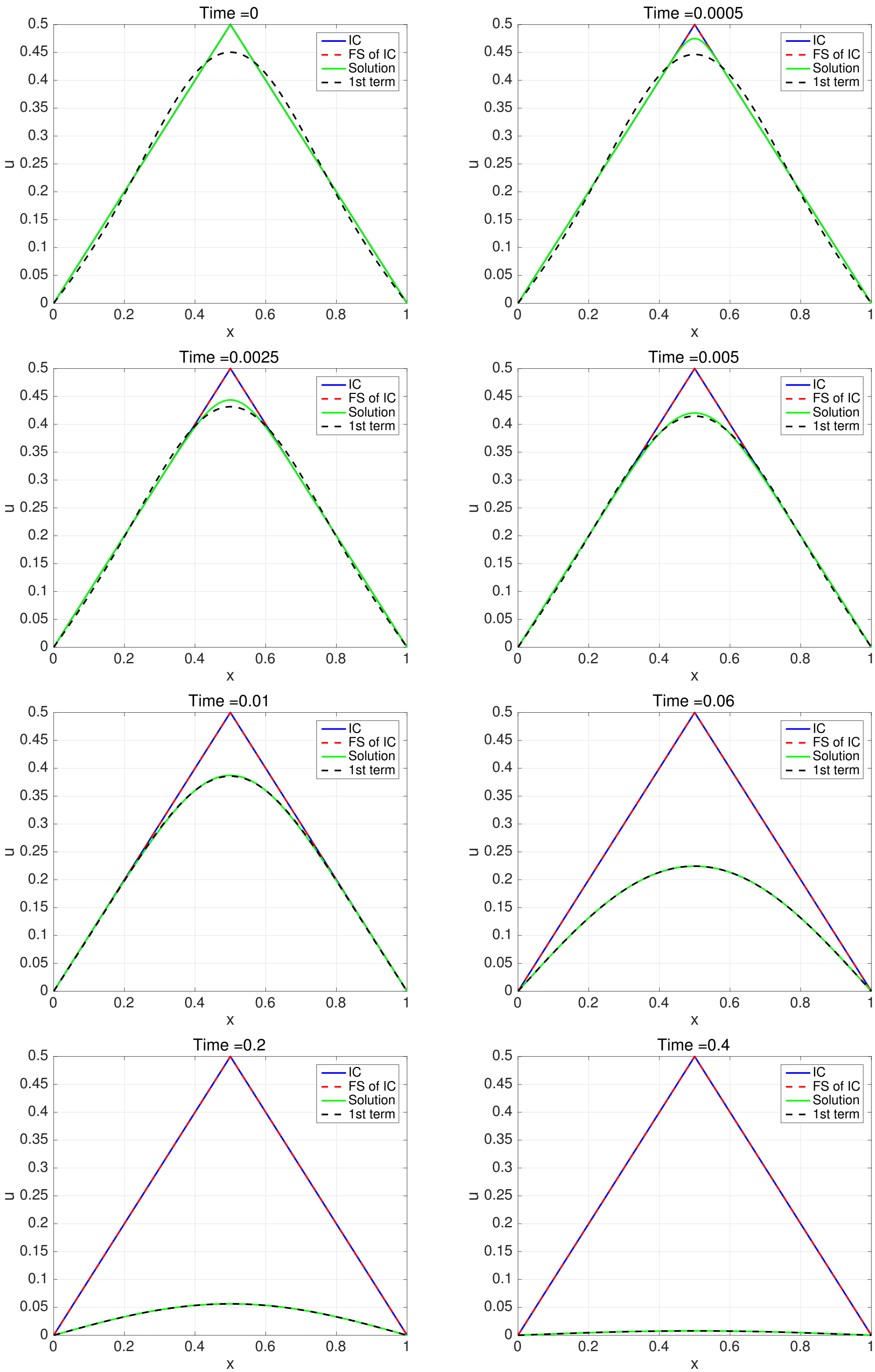
and since each v_n is CTS in x, so is u.

(B) Next Apply THM C to show that we can differentiate wrt x term by term (See HWK)

$$\frac{\partial u}{\partial x}(t,x) = \pi \sum_{n=1}^{\infty} n\, b_n \exp[-n^2\pi^2 t] \cos(n\pi x)$$

Applying Weierstrass M-Test once again see

$\frac{\partial u}{\partial x}$ is CTS in x. So $u \in C^1$, etc, $u \in C^\infty$.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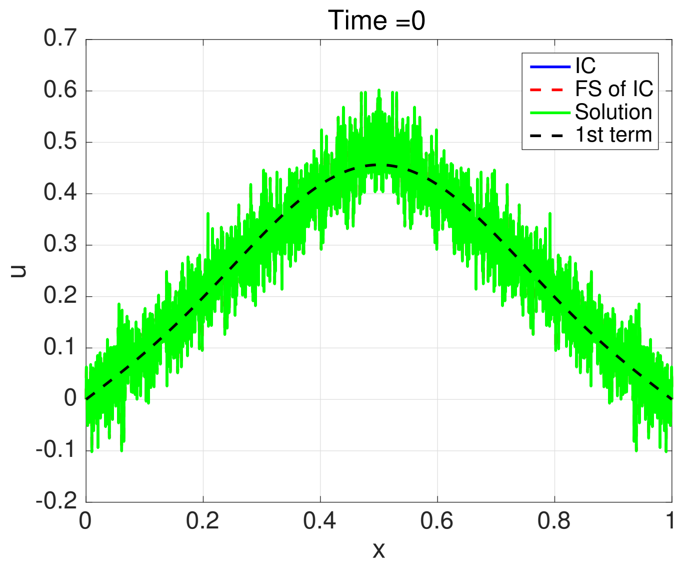
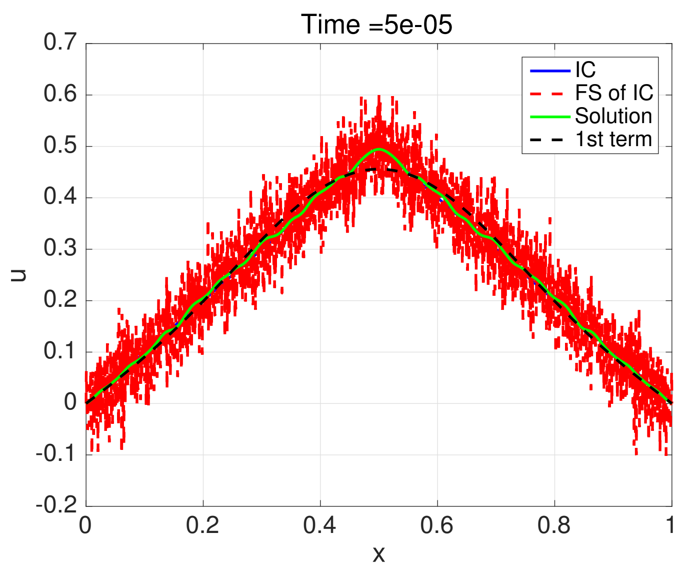
Time =0.0025
Time =0.005
Time =0.01
Time =0.06
Time =0.2
Time =0.4
IC
FS of IC
Solution
1st term
u
x



Time =0
IC
FS of IC
Solution
1st term
u
x



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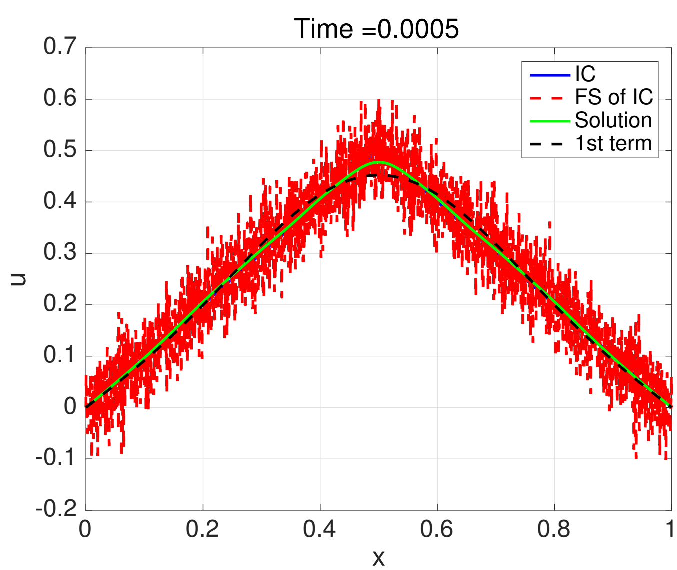
IC
FS of IC
Solution
1st term
u
x



Time =0.0005
IC
FS of IC
Solution
1st term
u
x