LECTURE 10 — CONVERGENCE OF FOURIER SERIES, II

THM [POINTWISE CONVERGENCE OF F.S.]

Let $f : [-\pi, \pi] \to \mathbb{C}$ be 2π-PERIODIC and Piecewise C^1.

Let
$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \qquad (1)$$

with
$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} \, dy \qquad (2)$$

Then $\forall x \in [-\pi, \pi]$

$$\lim_{n \to \infty} S_n(x) = \hat{f}(x) := \frac{1}{2}[f(x+) + f(x-)]$$

PF

PLUG (2) INTO (1) + USE LINEARITY OF $\int$:

$$S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{ik(x-y)} \, dy \qquad (3)$$

CLAIM
$$\sum_{k=-n}^{n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{1}{2}x)} \qquad (4)$$

PF OF CLAIM

$$\sum_{k=-n}^{n} e^{ikx} = e^{-inx} + e^{-i(n-1)x} + \dots + e^{-ix} + 1$$

$$+ e^{ix} + \dots + e^{inx}$$

$$= e^{-inx}\left[1 + e^{ix} + \dots + e^{2inx}\right]$$

$$= e^{-inx} \sum_{k=0}^{2n} \left(e^{ix}\right)^k$$

GEOMETRIC SERIES!!!

$$= e^{-inx} \sum_{k=0}^{2n} r^k \qquad \text{with } r = e^{ix}$$

$$= e^{-inx} \frac{1 - r^{2n+1}}{1 - r}$$

$$= e^{-inx} \frac{1 - e^{i(2n+1)x}}{1 - e^{ix}}$$

$$= \frac{e^{-inx} - e^{i(n+1)x}}{1 - e^{ix}}$$

$$= \frac{e^{ix/2}\left[e^{-i(n+\frac{1}{2})x} - e^{i(n+\frac{1}{2})x}\right]}{e^{+ix/2}\left[e^{-ix/2} - e^{ix/2}\right]}$$

$$= \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{2i} \; \frac{2i}{e^{ix/2} - e^{-ix/2}}$$

$$= \frac{\sin\left[(n+\frac{1}{2})x\right]}{\sin\left[\frac{x}{2}\right]}. \qquad \square$$

NEXT PLUG (4) INTO (3)

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,\frac{\sin[(n+\frac{1}{2})(x-y)]}{\sin[\frac{1}{2}(x-y)]}\,dy$$

$$\boxed{\begin{aligned} u &= y-x \\ du &= dy \end{aligned}}$$

$$= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} f(u+x)\,\frac{\sin[(n+\frac{1}{2})u]}{\sin(\frac{u}{2})}\,du$$

as sin IS ODD

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\,\frac{\sin[(n+\frac{1}{2})y]}{\sin(\frac{y}{2})}\,dy$$

as integrand is 2π-Periodic by (4)

CLAIM B

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{0}^{\pi} f(x+y)\,\frac{\sin((n+\frac{1}{2})y)}{\sin y/2}\,dy = f(x+) \quad (5)$$

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{0} f(x+y)\,\frac{\sin((n+\frac{1}{2})y)}{\sin y/2}\,dy = f(x-) \quad (6)$$

GIVEN CLAIM we conclude result

$$\lim_{n\to\infty} S_n(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right] \checkmark$$

$\boxed{\text{PF OF } 5}$ CLAIM C $\boxed{\frac{1}{\pi}\int_0^{\pi} \frac{\sin(n+\frac{1}{2})y}{\sin y/2}\, dy = 1}$ 7

PF By 4

$$\frac{1}{\pi}\int_0^{\pi} \underbrace{\frac{\sin(n+\frac{1}{2})y}{\sin y/2}}_{\text{EVEN.}}\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})y}{\sin y/2}\, dy$$

$$\overset{4}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-n}^{n} e^{iky}\, dy = 1$$

as only term $k=0$ has non zero integral.

By 5 and 7 sufficient to prove

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^{\pi} \frac{\left[f(x+y) - f(x^+)\right]}{\sin(y/2)} \sin\left[(n+\tfrac{1}{2})y\right] dy = 0.$$

CLAIM D

For each fixed x,

$$g(y) := \frac{f(x+y) - f(x^+)}{\sin(y/2)}$$

is piecewise CTS on $0 \le y \le \pi$.

PF

f is piecewise CTS, so only problem is at $y=0$.
and ~~by L'Hop~~

$$\lim_{y \to 0^+} g(y) = \lim_{y \to 0^+} \frac{f(x+y) - f(x^+)}{\sin(y/2)}$$

$$= 2 \lim_{y \to 0^+} \frac{f(x+y) - f(x^+)}{y} \quad \frac{y/2}{\sin(y/2)}$$

$$= 2 \quad f'(x^+) . \quad 1$$

as

$$f'(x^+) := \lim_{h \to 0^+} \frac{f(x+h) - f(x^+)}{h}$$

and

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

CLAIM E Let g be piecewise CB. Then

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi g(y)\sin\left(\left(n+\tfrac{1}{2}\right)y\right)dy = 0$$

PF This looks like Riemann-Lebesgue Lemma: which says if $\int_0^\pi |g^2(y)|\,dy < \infty$ Then

$$b_n = \frac{1}{\pi}\int_0^\pi g(y)\sin(ny)\,dy \to 0 \text{ as } n\to\infty$$

To deal with pesky $n+\frac{1}{2}$ factor:

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi g(y)\sin\left(\left(n+\tfrac{1}{2}\right)y\right)dy$$

$$= \lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi \left[g(y)\sin\left(\tfrac{1}{2}y\right)\right]\cos ny\,dy$$

$$+ \lim_{n\to\infty} \frac{1}{\pi}\int_0^\pi \left[g(y)\cos\left(\tfrac{1}{2}y\right)\right]\sin(ny)\,dy$$

$= 0 + 0$ by Riemann-Lebesgue applied to $g(y)\sin(y/2)$ and $g(y)\cos(y/2)$

D

$$\boxed{\text{SUMMARY OF PROOF IN CASE } f \text{ IS CTS AT } x}$$

$$S_n(x) = \sum_{|k|\le n} c_k e^{ikx} \quad \text{WITH} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-iky}\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \sum_{|k|\le n} e^{ik(x-y)}\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y) \frac{\sin[(n+\frac{1}{2})y]}{\sin[\frac{y}{2}]}\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x+y) - f(x)] \underbrace{\frac{\sin[(n+\frac{1}{2})y]}{\sin[\frac{y}{2}]}}_{K_n(y)}\, dy + f(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(x,y) \sin[(n+\tfrac{1}{2})y]\, dy + f(x)$$

$\longrightarrow 0 + f(x) = f(x)$ by RIEMANN-LEBESGUE

As ~~where~~ $G(x,y) = \dfrac{f(x+y) - f(x)}{\sin[\frac{y}{2}]}$ is piecewise CTS on $[-\pi, \pi]$

R-L SAYS IN $\boxed{\text{CASE } x = 0}$

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_n(y)\, dy \quad \text{as } \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\, dy = 1$$

The Sine Ratio Kernel

Let

$$K_n(y) \;=\; \frac{\sin[(n+\frac{1}{2})y]}{\sin[\frac{y}{2}]}. \tag{1}$$

Then for any continuous function, f,

$$f(0) \;=\; \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)K_n(y)\,dy. \tag{2}$$

This result is the special case of Claim B in Lecture 10, when f is continuous and $x = 0$. The idea is that

$$\lim_{n\to\infty} K_n(y) = \begin{cases} +\infty & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases} \tag{3}$$

Later in the course we will show that K_n is an approximation of the Dirac-δ distribution which has the property that

$$f(0) \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\delta(y)\,dy. \tag{4}$$

So as $n \to \infty$, in the integral we weight the value of f at $y = 0$ more and more compared to other values of f.

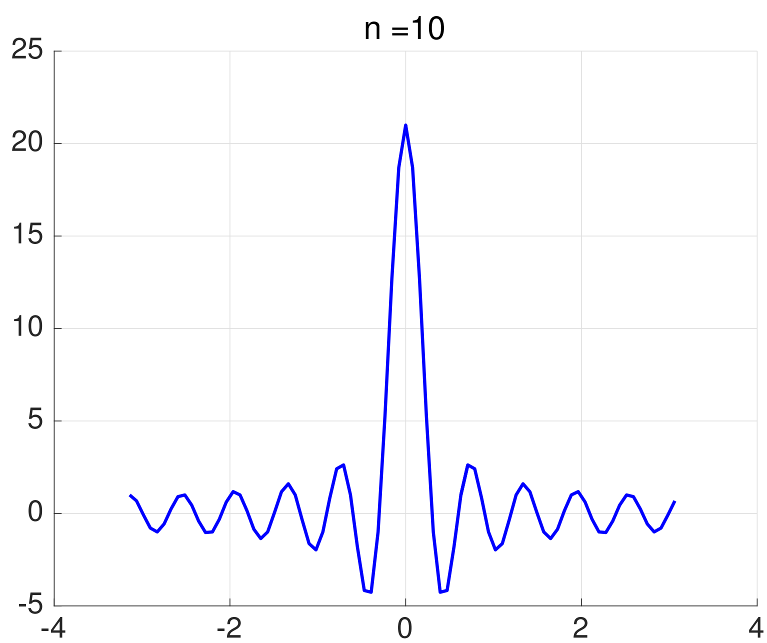

Figure 1: Plot of K_{10}.

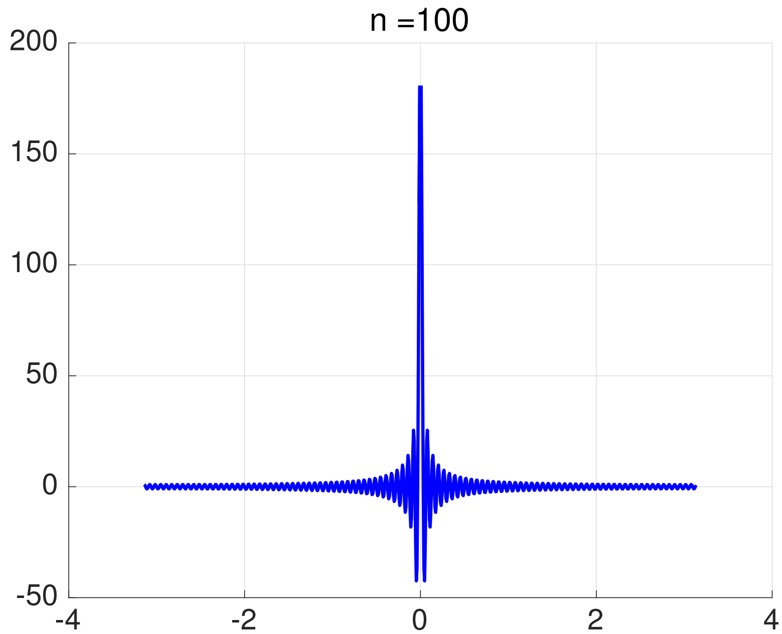

Figure 2: Plot of K_{100}.

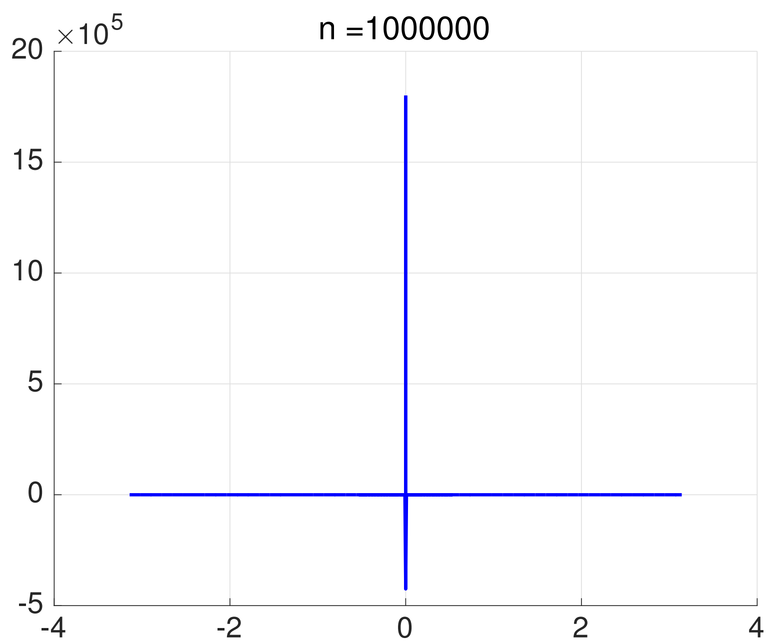

Figure 3: Plot of $K_{1,000,000}$.